PHYS 5040

Algebra and Topology in Physics

Spring 2021

Lecture 5

January 28

1 What does it mean for a quantum system to have symmetry group G?

quantum mechanics: Schrödinger eqn: $\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar} H|\psi\rangle$

time-independent

$H|\psi_E\rangle = E|\psi_E\rangle$

e-vectors

e-values

What does it mean to have symmetry group G?

$\exists$ set of unitary matrices $\{U(g)\}$ $g \in G$

form a group [under $\times$] isomorphic to G $[U(g_1 g_2) = U(g_1)U(g_2)]$

AND $[H, U(g)] = 0$ $\forall g \in G$.

$HU(g) = U(g)H$

$\rightarrow H = U(g)\,H\,U(g)^{-1}$

$= U(g)\,H\,U(g)^{\dagger}$

$i\hbar \frac{d}{dt} U(g)|\psi\rangle = U(g) H \overset{U^{-1}U}{\vee} |\psi\rangle$

$= \left(U(g) H U(g)^{-1}\right)\left(U(g)|\psi\rangle\right) = H\left[U(g)|\psi\rangle\right]$

2 Define a representation of a group G.

informally: representation R of group G is a set of matrices $R(g)$

$$\underbrace{R(g_1 g_2) = R(g_1) R(g_2)}$$

formally: Define group of invertible $n \times n$ [complex] matrices under multiplication: $GL(n, \mathbb{C})$

representation R as a homomorphism: $G \to GL(n, \mathbb{C})$

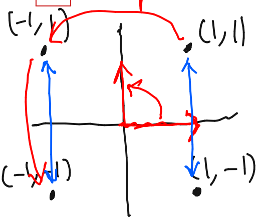
in quantum mechanics, there's a representation R of symmetry group G acts on Hilbert space

—AND, all matrices $R(g)$ are unitary...

unitary representation

dimension of R is n.

3 Write down a 2 dimensional representation of D_8.



$$D_8 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1} \rangle$$

r = rotation by 90°

$$R(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$R(r^2) = R(r)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R(r^4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R(1)$$

s = reflection:

$$R(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R(s^2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R(1)$$

$sr = r^3 s$?

$$R(sr) = R(s)R(r) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$R(r^3 s) = R(r^2)R(r)R(s) = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

4 Define an irreducible representation. Is it unique?

reducible representation: $R(g) = \begin{pmatrix} R_1(g) & 0 \\ 0 & R_2(g) \end{pmatrix}$ $\forall g \in G$ can happen, after any change of basis

$\underbrace{\quad}_{m}$ $\underbrace{\quad}_{n}$

then R is a reducible rep.

irreducible representation: NOT reducible (= irrep)

Next week: finite group G $\rightarrow$ generate (algorithm for) writing down ALL irreps

$\hookrightarrow$ up to "change of basis"/ similarity transforms:

$$S\,R(g)\,S^{-1}$$

$$S R(g_1) S^{-1} S R(g_2) S^{-1} = S R(g_1 g_2) S^{-1}$$

Character:

$$\chi^{(R)}(g) = \mathrm{tr}(R(g))$$

independent of S!

5 Find (all of the) irreducible unitary representations of $\mathbb{Z}_2$.

$\mathbb{Z}_2 = \{1, r\} \quad r^2 = 1$

$R(1)$ and $R(r)$ [unitary]

✓ $R(1)^2 = R(1)$

✓ $R(1)R(r) = R(r) = R(r)R(1)$

$R(r)^2 = R(1)$

$R(r)^2 = 1$ id matrix

if $R(r)$ had e-values $\mu_1, \dots, \mu_n$

$\mu_j^2 = 1 \quad \forall j$

so: $\mu_j = 1$ or -1

if $R(1)$ had eigenvalues $\lambda_1, \dots, \lambda_n$

$R(1)^2|j\rangle = R(1)|j\rangle$

$\lambda_j^2|j\rangle = \lambda_j|j\rangle$

$\lambda_j = 1$, ~~or 0~~.

identity matrix

$R(1) = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = 1$

summarize:

$R(1) = U\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}U^{-1}$

$R(r) = U\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}U^{-1}$

which reps irreducible?

go to "U basis": $R(1)$ & $R(r)$ diagonal.

- each 1×1 block on diagonal is an irrep...
- 2 irreps: $R(1) = 1$, $R(r) = \pm 1$

6 What is Schur's Lemma?

Schur's Lemma: Let R be an irrep of group of G. [$n \times n$ complex]

If $[H, R(g)] = 0 \quad \forall g \in G$, then $H = \underbrace{\lambda \times 1}_{\text{proportional to identity matrix}}$

w/o loss of generality

Proof (sketch): WLOG, assume H is diagonal.

$\forall g \in G, \quad (H R(g))_{ij} = (R(g) H)_{ij} \Rightarrow$

$H_{ii} R(g)_{ij} = R(g)_{ij} H_{jj}$

$0 = R(g)_{ij} (H_{ii} - H_{jj})$

either $H_{ii} = H_{jj}$

or $\underbrace{R(g)_{ij} = 0.}$

not ok?

if true [always...]

$H_{ii} = \lambda$ (goal!)

$$H = \left(\begin{array}{ccc|ccc} \lambda & & & 0 & & \\ & \lambda & & & & \\ & & \ddots & & & \\ \hline & & & \mu & & \\ & & & & \ddots & \\ & & & & & \mu \end{array}\right)$$

$\lambda \neq \mu$

$\forall g$

$$R(g) = \left(\begin{array}{ccc|cccc} & & & 0 & 0 & 0 & 0 \\ & R_1(g) & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & R_2(g) & & \\ 0 & 0 & 0 & & & & \end{array}\right)$$

this is reducible

if $H_{ii} \neq H_{jj}$, then $R(g)_{ij} = 0 \quad \forall g$

7 What does Schur's Lemma tell us about quantum systems with symmetry group G?

In quantum mech, symmetry $G \Rightarrow [H, U(g)] = 0$

"global" representation U ... in right basis,

$$U(g) = \begin{pmatrix} R_1(g) & 0 & 0 \\ 0 & R_2(g) & 0 \\ 0 & 0 & R_3(g) & \\ & & & \ddots \end{pmatrix}$$

irreps

if R_1 & R_2 are the "same" irrep, choice of how to make $U(g)$ block diagonal is not unique

$U = R_1 \oplus R_2 \oplus R_3 \oplus \cdots$

direct sum

[physics notation] rep $= 1 \oplus 3 \oplus 5$

1-dim   3-dim

by Schur's Lemma:

$$H = \begin{pmatrix} E_1 \ldots E_1 & 0 & 0 \\ 0 & E_2 \ldots E_2 & 0 \\ 0 & 0 & E_3 \ldots E_3 \end{pmatrix}$$

8 Consider a quantum system

$p = -i\hbar \frac{d}{dx}$

$$H = \frac{p^2}{2m} + V(x),$$

with parity operator $P\psi(x) = \psi(-x)$. When does $[H, P] = 0$? If obeyed, what can we conclude about the eigenvalues/eigenvectors of H?

P is unitary; $P^2\psi(x) = \psi(-(-x)) = \psi(x)$ $\quad P^2 = 1$

if $[H,P]=0$, then symmetry group $\{1, P\}$ isomorphic to $\mathbb{Z}_2$.

$[H,P]=0$? $\quad PHP^{-1} = PHP = P\left[\frac{p^2}{2m} + V(x)\right]P^{-1} = \frac{1}{2m}(\underbrace{PpP^{-1}}_{p})^2 + V(\underbrace{PxP^{-1}}_{-x})$

$= \frac{p^2}{2m} + V(-x) \overset{?}{=} H$

if $V(x) = V(-x)$

- irreps of $\mathbb{Z}_2$ corresponds to even/odd functions
- all e-vectors of H have to be even/odd
- b/c irreps of $\mathbb{Z}_2$ are 1-dim, there's no symmetry-enforced degeneracy