PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 6

February 2

1 Review represntations. State the Great Orthogonality Theorem. $g \in G$

QM: symmetry group correspond to unitaries $\{U(g)\}$

$$\underbrace{U(g_1)U(g_2) = U(g_1 g_2)}_{\text{(unitary) representation of } G} \quad \text{and} \quad [H, U(g)] = 0$$

how many different reps exist?

... up to change of basis

look at characters: $\chi^{(R)}(g) = \mathrm{tr}(R(g))$ ← or $D^{(R)}(g)$

$= \mathrm{tr}(S R(g) S^{-1})$

irreducible representation (irrep): can't be made block diagonal

Great Orthogonality Theorem: for any finite group G

$$\frac{1}{|G|} \sum_{g \in G} R(g)_{ij} \overline{Q(g)}_{kl} = \frac{1}{d_R} \delta_{RQ} \delta_{ik} \delta_{jl}$$

complex conjugation

$|G|$: # of elements in G

R, Q: $\boxed{\text{irreps}}$

d_R: dimension of rep R

2 Prove (part of) the Great Orthogonality Theorem.

Assume $R = Q$: show $\frac{1}{|G|}\sum_{g\in G} R(g)_{ij}\,\overline{R(g)}_{kl} = \frac{1}{d_R}\delta_{ik}\delta_{jl}$

Let X be arbitrary $d_R \times d_R$ matrix. Define $A = \frac{1}{|G|}\sum_{g\in G} R(g)^\dagger X R(g)$

$h \in G$

$$R(h)^\dagger A R(h) = \frac{1}{|G|}\sum_{g\in G} \overbrace{R(h)^\dagger R(g)^\dagger}^{=[R(g)R(h)]^\dagger} X R(g) R(h) = \frac{1}{|G|}\sum_{g\in G} R(gh)^\dagger X\, R(\underbrace{gh}_{g'})$$

$gh = g'$ $\quad$ $g = g'h^{-1}$

$= A$

or $A R(h) = R(h) A$

identity

Use Schur's Lemma: $A = \lambda \times \mathbb{1}$

e.g. $X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

sum over repeated indices

Pick $X_{ij} = \delta_{im}\delta_{jn}$

$$A_{ab} = \lambda\delta_{ab} = \frac{1}{|G|}\sum_{g\in G} \underbrace{\left(R(g)^\dagger\right)_{ai}}_{\overline{R(g)}_{ma}} \underbrace{X_{ij}}_{\delta_{im}\delta_{jn}} R(g)_{jb}$$

$\hookrightarrow = \frac{1}{d_R}\delta_{nm}\delta_{ab}$

$$= \frac{1}{|G|}\sum_{g\in G} \left(R(g)^\dagger\right)_{am} R(g)_{nb}$$

Assume $a = b$, sum over a:

$$\lambda\delta_{aa} = \underbrace{\lambda d_R}_{} = \frac{1}{|G|}\sum_{g\in G} \underbrace{R(g)^\dagger{}_{am} R(g)_{na}}_{R(g)_{na}R(g)^\dagger{}_{am} = \delta_{nm}} = \underbrace{\delta_{nm}}_{}$$

$\boxed{3}$ Review conjugacy classes; how do they relate to characters?

two group elements are conjugate: $g_1 \sim g_2 \iff \underline{g_1 = h^{-1} g_2 h}$
(if and only if) for some h

$$\chi^{(R)}(g_1) = \mathrm{tr}(R(g_1)) = \mathrm{tr}(R(h^{-1} g_2 h)) = \mathrm{tr}(R(h^{-1}) R(g_2) R(h))$$
$$= \mathrm{tr}(R(g_2) \cancel{R(h) R(h)^{-1}}) = \chi^{(R)}(g_2)$$

Hence, useful to organize group elements g into conjugacy classes $c = [g] = \{g, hgh^{-1}, \ldots\}$

G.O.T.:

$$\frac{1}{|G|} \sum_{g \in G} \overline{R(g)_{ij}}\, Q(g)_{kl} = \frac{1}{d_R} \delta_{RQ} \underbrace{\delta_{ik} \delta_{jl}}_{\substack{\delta_{ik}^2 = \delta_{ik} \\ \sum_{ik} \delta_{ik} = d_R}}$$

$$\overline{R(g)_{ii}} \to \overline{R(g)}_{ii} = \mathrm{tr}(R(g)^\dagger) = \overline{\chi^{(R)}(g)}$$

set $i=j$, $k=l$, sum over i,k

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi^{(R)}(g)}\, \chi^{(Q)}(g) = \delta_{RQ}$$

of g's in c

$$= \frac{1}{|G|} \sum_{c} n_c \overline{\chi^{(R)}(c)}\, \chi^{(Q)}(c)$$

4 Find all irreps of the group $\mathbb{Z}_n$. (cyclic group of order n)

Review: group of integers added (mod n)

also $\mathbb{Z}_n = \{1, r, r^2, \ldots, r^{n-1}\}$ with $r^n = 1$

Proposition: G is Abelian $\Leftrightarrow$ all complex irreps of G are 1-dimensional

Proof: $(\Leftarrow)$ if R is any rep of G,

$$R(g) = \begin{pmatrix} R_1(g) & 0 & 0 \\ 0 & R_2(g) & 0 \\ 0 & 0 & R_3(g) \end{pmatrix}$$

reducible!

if 2 diag mats D_1 & D_2: $D_1 D_2 = D_2 D_1$

$$R(g_1 g_2) = R(g_2 g_1)$$

if G weren't Abelian, $\exists$ rep where $R(g_1 g_2) \neq R(g_2 g_1)$

$(\Rightarrow)$: $g_1 g_2 = g_2 g_1 \Rightarrow [R(g_1), R(g_2)] = 0$ always.

$R(g_1)$ & $R(g_2)$ sim. diagonalizable

Irreps of $\mathbb{Z}_n$: Abelian $\Rightarrow$ 1-dim irrep

roots of unity:

$r^n = 1 \Rightarrow R(r)^n = R(1) = 1 \Rightarrow R(r) = e^{2\pi i \frac{k}{n}}, \; k = 0, 1, \ldots, n-1$

5 Give the character table of the group $\mathbb{Z}_n$.

representations (irreps)

	$k=0$	$k=1$	$\cdots$	$k=n-1$
1	1	1	$\cdots$	1
r	1	ω		ω^{-1}
r^2	1	ω^2		ω^{-2}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
r^{n-1}	1	$\omega^{n-1}=\omega^{-1}$		ω

Conjugacy classes

Abelian group

$hgh^{-1}=ghh^{-1}$

$=g$

fill in $\chi^{(R)}(c)$

$\omega=e^{2\pi i/n}$

6 Discuss the orthogonality properties of the character table.

$hgh^{-1}=1$	R_1 (irreps)	$\cdots$	R_n
$\{1\} \to C_1$	$\chi^{(R_1)}(C_1)$	$\cdots$	
conj classes $\vdots$	$\vdots$	$\ddots$	
C_n	$\chi^{(R_1)}(C_n)$		

Row orthogonality:

$$\sum_R \chi^{(R)}(c_i)\,\chi^{(R)}(c_j) = \frac{|G|}{n_{c_i}}\,\delta_{c_i c_j}$$

Therefore: # of rows = # of columns

irreps = # of conjugacy classes

look at conjugacy class [1]

$$|G| = \sum_R \left|\chi^{(R)}(1)\right|^2 = \sum_R d_R^2$$

Column Orthogonality:

$$\sum_c \underbrace{n_c}_{\#\text{ of elements}}\, \chi^{(R_i)}(c)\,\chi^{(R_j)}(c) = |G|\,\delta_{R_iR_j}$$

[follows from G.O.T.]

7 Find the conjugacy classes of D_8.

$$D_8 = \langle r, s \mid r^4 = 1,\ s^2 = 1,\ rs = sr^3 \rangle$$

$$= \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$$

$$(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$$

$$\boxed{[1] = \{1\}}$$

$[r] = \{r, r^3\}$. $[r]^2 = [r^2]$

$h r h^{-1} h r h^{-1} = h r^2 h^{-1}$

$$r^n r r^{-n} = r^{n+1-n} = r$$

$$(sr^n) r (sr^n)^{-1}$$
$$= sr^n r r^{-n} s$$
$$= srs = ssr^3 = r^3$$

$[r^2] = \{r^2\}$

$$sr^2s^{-1} = srs^{-1}srs^{-1}$$
$$= (r^3)^2 = r^6 = r^2$$

$[s] = \{s, r^2s\}$

$[rs] = \{rs, r^3s\}$

5 conj classes

8 Find the sole 2-dimensional irrep of D_8.

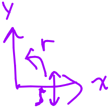
$$8 = d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

- not all $d_i = 1$: $8 > 5$
- but $d_i \le 2$: $8 < 9$
- $8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$

---

from last time:

$$r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

9 Find the character table of D_8.

irreps

conj classes	1	$1'$	$1''$	$1'''$	2
1	1	1	1	1	2
r^2	1	1	1	1	-2
r	1	-1	-1	1	0
s	1	-1	1	-1	0
rs	1	1	-1	-1	0

since 1 is identity matrix

from explicit construction

by row orth (1st & 2nd)

all must be a pure phase $e^{i\phi}$

in each column:

$a_1 a_2 = a_3$ since $R(rs) = R(r)R(s)$ $\rightarrow$ $a, b, c_3 = \pm 1$

a, b, c_1: $a_1^2 = b_1^2 = c_1^2 = 1$ $= \pm 1$

$s^2 = 1 \Rightarrow$ $a_2, b_2, c_2 = \pm 1$