PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 7

February 4

1 Review the character table for D_8. (finite group G)

conj classes	R_1	R_2	irreps $\cdots$	R_ℓ
c_1	$\chi^{(R_1)}(c_1)$	$\chi^{(R_2)}(c_1)$		
c_2	$\chi^{(R_1)}(c_2)$			
$\vdots$			$\ddots$	
c_ℓ				$\chi^{(R_\ell)}(c_\ell)$

$$g_1 \sim g_2 \iff g_1 = h g_2 h^{-1}, \quad h \in G$$

$$\chi^{(R)}(c) = \mathrm{tr}(R(g)) \quad \text{for any } g \in c$$

D_8	1 (trivial rep)	$1'$	$1''$	$1'''$	2 (vector)
$\{1\}$	1	1	1	1	2
$\{r^2\}$	1	1	1	1	-2
$\{r\}$	1	-1	-1	1	0
$\{s\}$	1	-1	1	-1	0
$\{rs\}$	1	1	-1	-1	0

row & column orthogonality

$$\sum_c n_c \, \overline{\chi^{(R_i)}(c)} \, \chi^{(R_j)}(c) = |G| \, \delta_{R_i R_j}$$

n_c: # of g in c

2 Describe how the Hilbert space of a realistic quantum system "decomposes into different irreps" of its symmetry group G.

quantum system w/ Hamiltonian H, symmetry group G

(reducible) representation U

$$[H, U(g)] = 0 \quad \forall g \in G$$

$$U = \underbrace{R_1 \oplus \cdots R_1}_{n_{R_1} \text{ times}} \oplus R_2 \oplus \cdots$$

direct sum

if you find right basis...

$$U(g) = \begin{pmatrix} R_1(g) & 0 & 0 & 0 & 0 \\ 0 & R_1(g) & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & R_2(g) & 0 \\ 0 & 0 & 0 & 0 & \ddots \end{pmatrix}$$

n_{R_1} times

$$H|n, R, k\rangle = E_{n,R}|n,R,k\rangle$$

irrep

$k = 1, \ldots, \dim(R)$

$n \quad \ell \quad m$

energy level

in this basis

Schur's Lemma:

$$H = \begin{pmatrix} E_{1,R_1}\mathbb{1} & 0 & 0 \\ 0 & E_{2,R_1}\mathbb{1} & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

e.g. hydrogen atom:

$$H|n\,\ell\,m\rangle = E_n|n\,\ell\,m\rangle$$

e-level

orbital angular mom.

z-component

3 Show how to project onto irrep R in a generic Hilbert space, even if the full irrep decomposition is not yet known.

Know $U(g)$, but don't know how to block diagonalize . . .

Goal: find a projector $P_{R'} \underbrace{|n, R, k\rangle}_{} = \delta_{RR'} |n, R, k\rangle$

Caution: formally, rep is $R(g)$ . . . often we'll use "irrep" to refer to vec. space $R(g)$ acts on

Claim: $P_R = \dfrac{\overset{\text{dimension}}{d_R}}{|G|} \sum_{g \in G} \overline{\chi^{(R)}(g)}\, U(g)$ projects onto irrep R

Proof: $P_{R'} |n, R, k\rangle = \dfrac{d_{R'}}{|G|} \sum_{g \in G} \overline{\chi^{(R')}(g)} \sum_{k'=1}^{d_R} R(g)_{kk'} |n R k'\rangle$

$\delta_{RR'} \delta_{kk'}$

G.O.T.:

$$\frac{d_{R'}}{|G|} \sum_{g \in G} \overline{R'(g)_{jj'}}\, R(g)_{kk'} = \delta_{RR'} \delta_{jk} \delta_{j'k'}$$

set $j = j'$ & sum: $\dfrac{d_{R'}}{|G|} \sum_{g \in G} \overline{\chi^{(R')}(g)}\, R(g)_{kk'} = \delta_{RR'} \sum_j \delta_{jk} \delta_{jk'} = \delta_{RR'} \delta_{kk'}$

$$P_{R'} |n R k\rangle = \sum_{k'} \delta_{RR'} \delta_{kk'} |n R k'\rangle = \delta_{RR'} |n R k\rangle$$

4 Given a generic representation R of a group G, how can we decompose it into irreps? Use characters.

Suppose
$$U = \underbrace{R_1 \oplus \cdots \oplus R_1}_{n_{R_1}} \oplus \underbrace{R_2 \oplus \cdots \oplus R_2}_{n_{R_2}} \oplus \cdots$$

n_{R_2} ↳ # of n in $|n R_2 k\rangle$

$$U(g) = \begin{pmatrix} \overbrace{\begin{array}{c|c|c} R_1(g) & 0 & 0 \\ \hline 0 & R_1(g) & 0 \\ \hline 0 & 0 & \ddots \end{array}}^{n_{R_1}\ \text{times}} & \begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array} \\ \begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} & \begin{array}{c|c} R_2(g) & 0 \\ \hline 0 & \ddots \end{array} \end{pmatrix}$$

$$\chi^{(U)}(g) = \mathrm{tr}(U(g))$$
$$= \sum \mathrm{tr}(\text{diagonal blocks})$$
$$= n_{R_1}\,\mathrm{tr}(R_1(g)) + \cdots$$
$$= \sum_{\text{irrep } R} n_R\, \chi^{(R)}(g)$$

$$\sum_{g\in G} \overline{\chi^{(R)}(g)}\,\chi^{(U)}(g) = \sum_C n_C\, \overline{\chi^{(R)}(C)}\,\chi^{(U)}(C)$$
$$= \sum_C \sum_{R'} \overline{\chi^{(R)}(C)}\, n_{R'}\, n_C\, \chi^{(R')}(C)$$
$$= \sum_{R'} |G|\,\delta_{RR'}\, n_{R'} = n_R |G|$$

5 Give a test for the irreducibility of a representation.

Suppose $U = R_1 \oplus \cdots \oplus R_2 \oplus \cdots$

is U irreducible or not?

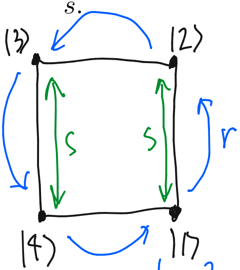
Evaluate $\sum_c n_c \overline{\chi^{(U)}(c)}\, \chi^{(U)}(c)$

$$= \sum_c \left(\sum_{R'} \overline{\chi^{(R')}(c)}\, n_{R'} \right) \left(\sum_{R''} n_{R''} \chi^{(R'')}(c) \right) n_c$$

by G.O.T. $= \sum_{R',R''} n_{R'} n_{R''} |G| \delta_{R'R''} = |G| \underbrace{\sum_R n_R^2}_{\text{integer} \geq 1}$

U is irreducible if and only if $\sum_R n_R^2 = 1$

6 Consider a quantum particle hopping on the 4 corners of a square, with a D_8-invariant Hamiltonian. Find the matrices corresponding to r and s.



$$U(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \chi^{(U)}(1) = d_U = 4$$

$$U(r^2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \chi^{(U)}(r^2) = 0$$

$$U(r) = \begin{array}{c c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \qquad \chi^{(U)}(r) = 0$$

$$U(rs) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \chi^{(U)}(rs) = 2$$

$$U(s) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \chi^{(U)}(s) = 0$$

7 How does the Hilbert space decompose into irreps? Determine this explicitly, using characters.

trivial rep, vector

n_c	D_8	1	$1'$	$1''$	$1'''$	2
1	$\{1\}$	1	1	1	1	2
1	$\{r^2\}$	1	1	1	1	-2
2	$\{r\}$	1	-1	-1	1	0
2	$\{s\}$	1	-1	1	-1	0
2	$\{rs\}$	1	1	-1	-1	0

$$\chi^{(U)} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}\begin{array}{c} 1 \\ r^2 \\ r \\ s \\ rs \end{array}$$

$$n_1 = \frac{1}{8}\sum_c n_c \overline{\chi^{(1)}(c)}\,\chi^{(U)}(c)$$

$$= \frac{1}{8}\left[1\cdot 1\cdot 4 + 1\cdot 1\cdot 0 + 2\cdot 1\cdot 0 + 2\cdot 1\cdot 0 + 2\cdot 1\cdot 2\right] = \underline{1 = n_1}$$

$\underline{n_{1'} = 1 \qquad n_2 = 1}$

$n_{1''} = n_{1'''} = 0$

$$\boxed{U = 1 \oplus 1' \oplus 2}$$

8 Determine the most general Hamiltonian consistent with symmetry, and discuss its degeneracy.

$$H = \begin{pmatrix} a & b & c & b \\ b & a & b & c \\ c & b & a & b \\ b & c & b & a \end{pmatrix}$$

Since $[H, U(r)] = 0$

eigenvectors of $H \leftrightarrow$ eigenv's of $U(r)$

$$U(r) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

eigenvectors of $U(r)$

$$= \begin{pmatrix} 1 \\ e^{i\phi} \\ e^{2i\phi} \\ e^{3i\phi} \end{pmatrix} \qquad e^{4i\phi} = 1$$

eigenvalue $e^{i\phi} = \{1, -1, i, -i\}$

eigenvectors of H

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \underbrace{\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}}$$

$$\boxed{a+c+2b \qquad a+c-2b \qquad a-c}$$ eigenvalues

irreps $1 \oplus 1' \oplus 2 = U$

9 Use projectors to show how the state $|1\rangle$ decomposes into basis vectors associated with each irrep.

$$P_1|1\rangle = \frac{1}{8}\sum_{g\in D_8} \overline{\chi^{(1)}(g)}\, U(g)|1\rangle$$

$$= \frac{1}{8}\sum_{g\in D_8} 1\cdot\Big[\overset{1}{|1\rangle} + \overset{r}{|2\rangle} + \overset{r^2}{|3\rangle} + \overset{r^3}{|4\rangle} + \overset{s}{|2\rangle} + \overset{rs}{|3\rangle} + \overset{r^2s+r^3s}{|4\rangle + |1\rangle}\Big]$$

$$= \frac{1}{4}\big[|1\rangle + |2\rangle + |3\rangle + |4\rangle\big]$$

$$P_{1'}|1\rangle = \frac{1}{8}\sum_{g\in D_8}\big[|1\rangle - |2\rangle + |3\rangle - |4\rangle - |2\rangle + |3\rangle - |4\rangle + |1\rangle\big]$$

$$= \frac{1}{4}\big[|1\rangle - |2\rangle + |3\rangle - |4\rangle\big]$$

$$P_2|1\rangle = \frac{2}{8}\sum_{g\in D_8}\Big[2\overset{1}{|1\rangle} - 2\overset{r^2}{|3\rangle}\Big] = \frac{1}{2}\big[|1\rangle - |3\rangle\big]$$