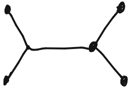
PHYS 5040
Algebra and Topology in Physics
Spring 2021

Lecture 8

February 9

$\boxed{1}$ What are the irreps of a product group $G \times H$?



$|x\rangle$ [position] $\otimes$ $|s\rangle$ [spin]

full wave function $|\psi\rangle = \sum \psi_{xs} \overbrace{|x\rangle \otimes |s\rangle}^{|x,s\rangle}$

G ~ spatial reflection $[|x\rangle]$: representation $U(g)|x\rangle = \sum_{x'} U(g)_{xx'} |x'\rangle$

H ~ spin rotation $[|s\rangle]$ rep $V(h)|s\rangle = \sum_{s'} V(h)_{ss'} |s'\rangle$

direct product group $G \times H$: elements $(\overset{\in G}{g}, \overset{\in H}{h})$: $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$

representation of $G \times H$; $(g,h) \circ |x\rangle \otimes |s\rangle = U(g)|x\rangle \otimes V(h)|s\rangle$

denote full 12×12 matrix [representation] $U(g) \otimes V(h)$

$$(U \otimes V)|x,s\rangle = \sum U_{xx'} V_{ss'} |x', s'\rangle$$

irreps of $G \times H$? Always $(\underset{\text{irrep } G}{R}, \underset{\text{irrep } H}{Q})$: $R(g) \otimes Q(h)$

$$\chi^{(R,Q)}(g,h) = \chi^{(R)}(g)\, \chi^{(Q)}(h)$$

$\boxed{2}$ What happens to the representations of G if we restrict to a subgroup $K \le G$?

$\overset{\downarrow \in K}{U(1)}, U(g_1), \overset{\downarrow \in K}{U(g_2)} \ldots$

$U(g_1 g_2) = U(g_1)U(g_2)$

$U(k_1 k_2) = U(k_1)U(k_2)$

- Let U be a [reducible?] rep of G

  ↳ [reducible?] rep of K

- usually, irrep R in group $G \longrightarrow$ reducible rep. of subgroup K

$$\underset{\text{irrep } G}{R} = \underbrace{Q_1 \oplus \cdots \oplus Q_\ell}_{\text{irreps of } K}$$

$[H', U(g)] = 0$ if and only if $g \in K$

- Application [theme]: $H = \underset{\text{sym } G}{H_0} + \delta \underset{\text{perturbation: sym group } K.}{H'}$

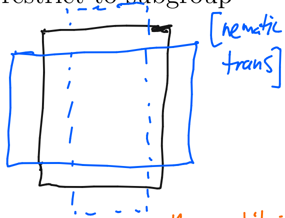
$$H_0 = \begin{pmatrix} \begin{array}{c|c|c} & \overbrace{\qquad\qquad}^{R} & \\ E_0 & & \\ \hline & E_1 \ddots E_1 & \\ \hline & & E_2 \ldots E_2 \end{array} \end{pmatrix}$$

$$H' = \begin{pmatrix} \begin{array}{c|cc:cc|c} & \multicolumn{2}{c:}{\overbrace{\qquad}^{Q_1}} & \multicolumn{2}{c|}{\overbrace{\qquad}^{Q_2}} & \\ \hline & E_1+\delta\varepsilon_1 & 0 & 0 & 0 & \\ & 0 & E_1+\delta\varepsilon_1 & 0 & 0 & \\ \hdashline & 0 & 0 & E_1+\delta\varepsilon_2 & 0 & \\ & 0 & 0 & 0 & E_1+\delta\varepsilon_2 & \\ \hline & \multicolumn{4}{c|}{\underbrace{\qquad\qquad\qquad}_{R}} & \end{array} \end{pmatrix}$$

3 What happens to the representations of D_8 if we restrict to subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$?

$D_8 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$

$\hookrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, r^2, s, r^2s\}$



D_8	1	1′	1″	1‴	2
1	1	1	1	1	2
r^2	1	1	1	1	−2
r, r^3	1	−1	−1	1	0
s, r^2s	1	−1	1	−1	0
rs, r^3s	1	1	−1	−1	0

D_8	$\mathbb{Z}_2 \times \mathbb{Z}_2$
1	R_{++}
1′	R_{+-}
1″	R_{++}
1‴	R_{+-}
2	$R_{-+} \oplus R_{--}$

character orthogonality:

$$n_{R_{++}} = \frac{1}{4} \sum_{g \in \mathbb{Z}_2 \times \mathbb{Z}_2} \chi^{(R_{++})}(g)\, \chi^{(2)}(g)$$

$$= \frac{1}{4}\left[2 \cdot 1 + (-2) \cdot 1 + 0\right]$$

$$= 0$$

$$n_{R_{-+}} = \frac{1}{4}\left[2 \cdot 1 + (-2)(-1) + 0\right]$$

$$= 1$$

$\mathbb{Z}_2$	+	−
1	1	1
s	1	−1

$\mathbb{Z}_2 \times \mathbb{Z}_2$	R_{++}	R_{-+}	R_{+-}	R_{--}
1	1	1	1	1
r^2	1	−1	1	−1
s	1	1	−1	−1
r^2s	1	−1	−1	1

$\boxed{4}$ In D_8, what is $\mathbf{2} \otimes \mathbf{2}$?

Example: 2 of D_8 corresponds to vectors: (x, y), (J_x, J_y), (E_x, E_y)

$2 \otimes 2$: $a_i b_j \longrightarrow 2 \times 2$ matrix M_{ij}

under action of D_8: if 2 was $V(g)_{ij} a_j = g \cdot a_i$

in $2 \otimes 2$: $(V \otimes V)_{ij,kl} M_{kl} = V_{ik} V_{jl} M_{kl}$

how many 2's in $2 \otimes 2$

$$n_2 = \frac{1}{8} \sum_c n_c \overline{\chi^{(2)}(c)} \underbrace{\chi^{(2\otimes 2)}(c)}_{= \chi^{(2)}(c)^2}$$

error in class

$$= \frac{1}{8}\left[1 \cdot 2 \cdot 2^2 + 1 \cdot (-2)(-2)^2\right]$$

$$= 0$$

$$n_1 = \frac{1}{8} \sum_c n_c \overline{\chi^{(1)}(c)} \chi^{(2\otimes 2)}(c)$$

$$= \frac{1}{8}\left(1 \cdot 1 \cdot 2^2 + 1 \cdot 1 \cdot (-2)^2\right) = 1$$

Keep on going:

$$2 \otimes 2 = 1 \oplus 1' \oplus 1'' \oplus 1'''$$

5 Consider a 2d crystal with D_8 symmetry. What is the most general possible conductivity tensor?

Ohm's Law: $\frac{\Delta V}{R} = I$

$$\sigma_{ij} E_j = J_i$$

general $\sigma_{ij} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

what are a,b,c,d in 1 irrep?

$$P_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{8} \sum_{g \in D_8} \overset{1}{\cancel{\chi^{(1)}}}(g)\, g \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$g \cdot M_{ij} \rightarrow V(g)_{ik} V(g)_{jl} M_{kl}$$

After calculation:

$$P_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a+d}{2} & 0 \\ 0 & \frac{a+d}{2} \end{pmatrix}$$

Need σ_{ij} is D_8-invariant,

$$\underset{\uparrow \in D_8}{g} \cdot \sigma_{ij} = \sigma_{ij}$$

σ_{ij} needs to be in 1 [trivial representation]

Conclude that

$$\sigma_{ij} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} = \sigma_0 \delta_{ij}$$

Also get this from Schur's Lemma! $[\sigma, V(g)] = 0$

- J & E are in irrep 2.

6 Decompose a 2×2 tensor into the various irreps of D_8.

$$\begin{tikzcd}
2\otimes 2 = & 1 \arrow[d] & \oplus & 1' \arrow[d] & \oplus & 1'' \arrow[d] & \oplus & 1''' \arrow[d] \\
& \begin{pmatrix}1&0\\0&1\end{pmatrix} & & \begin{pmatrix}0&1\\1&0\end{pmatrix} & & \begin{pmatrix}1&0\\0&-1\end{pmatrix} & & \begin{pmatrix}0&-1\\1&0\end{pmatrix}
\end{tikzcd}$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} = \frac{a+d}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{a-d}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$+ \frac{b+c}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \frac{c-b}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

7 Consider a 2d crystal with D_8 symmetry. What is the most general possible elasticity tensor?

elasticity: $\underline{\text{Hooke's Law}}$

both τ & s are symmetric

$$-F = k \Delta x$$

continuum

$$\tau_{ij} = \lambda_{ij,kl} \, s_{kl}$$

(stress tensor)

[force (i-comp) acting in the j-direction/area]

(strain tensor)

$$\Delta_k x_l + \Delta_l x_k$$

How many different elastic coeff [$\lambda_{ij,kl}$ eigenvalues]

λ must be D_8 invariant, By Schur's Lemma:

strain tensors

$$1 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftarrow \delta_{ij}, \quad 1' : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftarrow \sigma^x_{ij}$$

$$1'' : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \leftarrow \sigma^z_{ij}$$

$$\lambda_{ij,kl} = \lambda_1 \delta_{ij} \delta_{kl} + \lambda_2 \sigma^x_{ij} \sigma^x_{kl} + \lambda_3 \sigma^z_{ij} \sigma^z_{kl}$$

e.g. $\delta_{kl} M_{kl} = tr(M)$