PHYS 5040: Algebra and Topology in Physics **Spring 2021**

Practice Exam

- **Prove/show** means to provide a mathematically rigorous proof. **Argue/describe/explain why** means a non-rigorous (but convincing) argument is acceptable.

20 **Problem 1:** What are the following groups?

1.1. $\pi_3(\mathrm{T}^3)$

1.2. $\pi_1(\mathrm{T}^3)$

1.3. $\pi_1(\mathrm{S}^5)$

1.4. $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2)$

1.5. $\mathrm{H}_3(\mathrm{S}^2 \times \mathrm{S}^1)$

20 **Problem 2 (Center of a group):** Define the **center** of a group G to be the set of all elements that commute with all others:

$$\mathrm{Z}(G) := \{h \in G : \forall g \in G, gh = hg\}. \tag{1}$$

2.1. Show that the center is a subgroup: $\mathrm{Z}(G) \leq G$.

2.2. What is $\mathrm{Z}(\mathbb{Z}_8)$?

2.3. What is $\mathrm{Z}(\mathrm{D}_8)$?

2.4. What is $\mathrm{Z}(\mathrm{SU}(n))$?[1]

20 **Problem 3:** Consider the $(2,0)$ irrep of the group $\mathrm{SU}(3)$.

3.1. What type of tensor does this irrep correspond to?

3.2. What is the dimension of this irrep?

3.3. Evaluate $(2,0) \otimes (2,0)$; express the result in terms of irreps of $\mathrm{SU}(3)$.

3.4. Suppose that we consider an $\mathrm{SU}(2)$ subgroup of $\mathrm{SU}(3)$. In the fundamental representation of $\mathrm{SU}(3)$, this subgroup can be defined as

$$\mathrm{SU}(2) = \left\{ M \in \mathrm{SU}(3) : M \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{2}$$

Determine how the irrep (2,0) decomposes into irreps of $\mathrm{SU}(2)$.

[1] *Hint:* Think about $\mathrm{SU}(n)$ in the defining representation of complex $n \times n$ matrices, and use Schur's Lemma.

20 **Problem 4 (D-branes and flux threading):** In string theory, Dp-branes are objects that are extended in $p+1$ spacetime dimensions. Dp-branes can be thought of as generalizations of particles (D0 branes) to higher dimensional objects, extending in p spatial dimensions. The "worldvolume" of a Dp brane is $p+1$ dimensional because it sweeps out a trajectory in time as well.

4.1. Suppose that we would like to couple these Dp-branes to some kind of generalized electromagnetism. This means that we need a gauge field B which is a q-form, such that

$$\text{phase factor} = \exp\left[\mathrm{i}\int_{\text{worldvolume}} B\right] \tag{3}$$

is well-defined. What must the value of q be?

4.2. Given a gauge field B, we can generalize the notion of the Maxwell tensor $F = \mathrm{d}A$ of electromagnetism by defining $C = \mathrm{d}B$. In type IIB string theory, there is a flux C_5 which is a 5-form. Inverting the chain of logic from before, deduce that there must be a Dp-brane in type IIB string theory – what is the value of p?

4.3. One possibility suggested by string theory is that our universe may actually have 9 spatial dimensions; however, 6 of them will be *compact*. Hence our universe would have spatial geometry $\mathbb{R}^3 \times M^6$, where M^6 denotes a 6-dimensional compact manifold. The compactness of M^6 can be stabilized by fluxes of p-form flux through non-trivial p-dimensional cycles S of M^6. Through which of the following manifolds can we find a non-trivial S through which to thread 5-form flux C_5: $\mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$, $\mathrm{S}^3 \times \mathrm{T}^3$, $\mathrm{S}^1 \times \mathrm{S}^5$, S^6, T^6? Here T^n denotes the n-dimensional torus.

Problem 5: Consider an incompressible fluid in two spatial dimensions, obeying

$$\nabla \cdot \mathbf{v} = 0 \tag{4}$$

where $\mathbf{v}$ denotes the velocity field of the fluid.

15 **5A:** Suppose that we study this fluid inside of the domain

$$A = \{(x, y) \in \mathbb{R}^2 : \; a^2 \le x^2 + y^2 \le b^2\}. \tag{5}$$

5A.1. Show how to express A in terms of a CW complex.

5A.2. Show, by explicit computation of the (co)homology groups using the CW complex above, that the de Rham cohomology group $\mathrm{H}^1(A) = \mathbb{R}$. Explain why this answer is reasonable.

5A.3. Find the non-trivial 1-form $\chi = \chi_x \mathrm{d}x + \chi_y \mathrm{d}y$ associated to $\mathrm{H}^1(A)$.[2]

10 **5B:** What are the physical consequences of this non-trivial first cohomology group? We can associate the velocity field $\mathbf{v}$ with a 1-form

$$v = v_x \mathrm{d}x + v_y \mathrm{d}y. \tag{6}$$

5B.1. Write (4) in terms of differential forms and operations on them.

5B.2. Find the most general solution to (4) in this domain.

5B.3. What is the physical implication of the non-trivial cohomology of A?

[2] *Hint:* You can work in polar coordinates in principle, but if you are not savvy with geometry, you might get tripped up later on. The safest way (albeit slightly more tedious) to proceed is to stick with rectangular coordinates.

Problem 6: Consider two interacting particles hopping between three sites $|1\rangle$, $|2\rangle$, $|3\rangle$, which we can think of as the corners of an equilateral triangle. The total Hamiltonian is

$$H = h(a) \otimes 1 + 1 \otimes h(b) + V \tag{7}$$

where the single-particle hopping Hamiltonians $h(a)$ are defined via (here a denotes a hopping strength)

$$h(a) = a|1\rangle\langle 2| + a|2\rangle\langle 1| + a|3\rangle\langle 2| + a|2\rangle\langle 3| + a|1\rangle\langle 3| + a|3\rangle\langle 1|, \tag{8}$$

and the two-body interaction is

$$V = V_0 \left(|11\rangle\langle 11| + |22\rangle\langle 22| + |33\rangle\langle 33|\right). \tag{9}$$

10 **6A:** Consider the 9 dimensional representation U of the permutation group S_3, defined as

$$U(\sigma)|ij\rangle = |\sigma(i)\sigma(j)\rangle. \tag{10}$$

Show that for all σ, $[H, U(\sigma)] = 0$. Thus our Hamiltonian has S_3 symmetry.

20 **6B:** What are the consequences of S_3 symmetry for this quantum mechanical problem?

- 6B.1. Find the eigenvalues of H when $V_0 = 0$ (i.e. the non-interacting limit). Describe the degeneracies of H.
- 6B.2. Now, decompose U into irreps of S_3. How many copies of $\mathbf{1}$, $\overline{\mathbf{1}}$, and $\mathbf{2}$ are there in the Hilbert space?
- 6B.3. Which energy levels of H_0 (when $V_0 = 0$) correspond to which irreps of S_3?
- 6B.4. What do you predict happens to the energy levels of the Hamiltonian when $V_0 \neq 0$ (the interaction is turned on)?
- 6B.5. Check your prediction by numerically calculating the eigenvalues of H. Take "generic" values of a, b and V_0 to avoid any accidental degeneracies.