PHYS 5210: Graduate Classical Mechanics **Fall 2022**

Exam

Due: December 12 at 11:59 PM. Submit on Canvas.

You are allowed to refer to any course materials (including posted solutions), any books, and the Internet (e.g. Wikipedia, papers). **Do not collaborate** with any human; **do not solicit help** via PhysicsForums, Chegg, Quora or any similar website. You may ask the instructor alone for help in the form of clarifying questions. Please cite (in any reasonable way) any online resources you have used.

Problem 1 (Flux quantization): In quantum mechanics, a small ring of radius R made out of superconducting material will expel magnetic field lines out of the center if the magnetic field is small enough. We can understand this effect using (mostly) classical mechanics. Imagine a single particle of charge q, whose configuration space is S^1. The coordinate $\theta \sim \theta + 2\pi$ is an angular coordinate.

15 **A:** Postulate we have rotation invariance: L is invariant under $\theta(t) \to \theta(t) + \epsilon$ for any constant ϵ.

- A1. Explain why, keeping at most two time derivatives in the Lagrangian, the most general equations of motion we can write down come from (for some constant I)

$$L = \frac{I}{2}\dot{\theta}^2. \tag{1}$$

- A2. What are the equations of motion for this theory? What are their solutions?
- A3. Use Noether's Theorem to deduce the conserved quantity associated with rotation symmetry.

15 **B:** Let B be the uniform field strength inside the center of the ring. The Lagrangian then becomes modified (you don't need to show it) to

$$L = \frac{I}{2}\dot{\theta}^2 + \frac{qBR^2}{2}\dot{\theta}. \tag{2}$$

- B1. Does the B term change the equations of motion? Why or why not?
- B2. Now, consider the trajectory $\theta(t) = \omega t$ for $0 < t < 2\pi/\omega$. Evaluate the action S on this trajectory.
- B3. In the limit $\omega \to 0$, the resulting action (in a superconductor) should vanish (since a small current can flow freely). More precisely, in a quantum theory, we want $\mathrm{e}^{\mathrm{i}S/\hbar} = 1$. Argue that this criterion is only satisfied if

$$B = n \times \frac{2\hbar}{qR^2}, \tag{3}$$

  where n is an integer. Therefore, the magnetic field is **quantized**.

This effect, known as flux quantization, is readily seen in experiments on small superconducting loops. If the field strength is much smaller than the value above, the magnetic field will be expelled outside of the center of the ring (by microscopic currents flowing in the superconductor).

Problem 2: Consider a charged particle of mass m and charge q, moving in the electric potential of a static dipole. For simplicity in this problem, restrict to the xy plane, so that the electric potential is

$$\Phi(x,y) = \frac{\alpha}{4\pi\epsilon_0}\frac{y}{(x^2+y^2)^{3/2}}. \tag{4}$$

15 **A:** Let us begin with the Lagrangian mechanics of this system.

A1. Use results from class – or elsewhere – to write down the Lagrangian L describing *non-relativistic* motion of the charge q.

A2. Convert to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Find expressions for $\dot{x}$ and $\dot{y}$ in terms of the r, θ variables. Then show that in polar coordinates,

$$L = \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{q\alpha}{4\pi\epsilon_0}\frac{\sin\theta}{r^2}. \tag{5}$$

10 **B:** Convert L to a Hamiltonian $H(r,\theta,p_r,p_\theta)$ by doing the Legendre transform. Show that

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{q\alpha}{4\pi\epsilon}\frac{\sin\theta}{r^2}. \tag{6}$$

10 **C:** Now, let us employ the Hamilton-Jacobi approach.

C1. Write down the Hamilton-Jacobi equation for $S(r,\theta,t)$.

C2. Show that it can be solved using the separation of variables ansatz $S = W_r(r) + W_\theta(\theta) - Et$. "Solve" for W_θ and W_r – though leave your answers in terms of integrals (that you don't need to evaluate). The goal of this problem is not to do the painful algebra, but rather to recognize why and how this problem might be solved using the Hamilton-Jacobi method.

Problem 3: Consider the discrete map (here $r > 0$)

$$x_{n+1} = -rx_n + (1+r)x_n^3. \tag{7}$$

10 **A:** Let us begin with some analytic results.

A1. We want the dynamics to maintain $|x_n| \le 1$. Confirm that this is the case so long as $r < 3$.

A2. Find the fixed points. Deduce (for any value of r) whether they are stable or unstable.

10 **B:** Numerically simulate this map.

B1. Plot, in the (r,x) plane, all the possible points reached by this discrete mapping at large n. This is called the attractor.[1]

B2. Based on your plot, argue that there is a transition to chaos with increasing r. Compare the transition to chaos between this map, and the logistic map. Is it similar? Why or why not?

[1] *Hint:* We looked at this plot in Lecture 36, and the `Mathematica` notebook used to generate the attractor there can be used to generate it here, with modifications to only one or two lines of code!

Problem 4 (Burgers' equation): Burgers' equation is a fluid dynamical equation for a velocity field $v(x,t)$ in one spatial dimension:

$$\partial_t v + v\partial_x v = 0. \tag{8}$$

Remarkably, this theory actually comes from a Lagrangian!

15 **A:** Show that if we define $v = \partial_x \phi$, then (8) is the equation of motion for

$$\mathcal{L} = -\frac{1}{2}\partial_x\phi\partial_t\phi - f(\partial_x\phi), \tag{9}$$

for a special choice of the function f. What is it?

10 **B:** Identify three continuous symmetries of this theory (9) that follow from simple invariances of the action (e.g. similar to things we have seen in this class before). Then use Noether's Theorem to identify three conserved currents. Check explicitly that they are all conserved.

5 **C:** Argue that in fact, there are an infinite number of conservation laws.[2] Find continuous symmetries of the problem that generate these conservation laws, and show that the action is invariant under your proposed continuous symmetries.

10 **Problem 5:** Consider the canonical symplectic form on symplectic manifold (i.e. phase space) $\mathbb{R}^4$, spanned by (x, y, p_x, p_y): $[x, p_x] = 1$, $[x, y] = 0$, etc. (cf Lecture 24).

1. Find three functions of the phase space coordinates: A, B, C, that obey the Poisson brackets

$$[A, B] = 0, \tag{10a}$$
$$[A, C] = A, \tag{10b}$$
$$[B, C] = -B. \tag{10c}$$

   Do not use complex coefficients in your answer – you must find 3 real-valued functions of phase space coordinates that obey (10). Also, A and B cannot be functions of each other (e.g. do not use $A = B^2$).

2. Find a Hamiltonian $H \neq 0$, which is well-defined everywhere on phase space, that obeys[3]

$$[H, A] = [H, B] = [H, C] = 0. \tag{11}$$

3. Is this theory integrable? If so, try to find exact solutions to Hamilton's equations for arbitrary initial conditions.

[2] *Hint:* In retrospect, can you argue directly from (8) that your three conserved currents from **B** are conserved? If so, then find even more!

[3] *Note:* I have found a construction where this indeed works. So if your answer to 1 obeys (10), but you cannot find an H which isn't singular, then try to pick a different A, B, C!