PHYS 5210: Graduate Classical Mechanics **Fall 2022**

Homework 1

Due: August 29 at 11:59 PM. Submit on Canvas.

Problem 1 (Square potential): Consider a two-dimensional particle of mass m moving in a generic potential $V(x, y)$. Suppose that $V(x, y) \geq 0$, and that the unique minimum of V is located at $x = y = 0$.

10 **A:** Based on principles of effective theory, if the particle is moving very close to $x = y = 0$, explain why you can assume that

$$V(x, y) = \frac{Ax^2 + Bxy + Cy^2}{2} + \cdots . \tag{1}$$

10 **B:** Now suppose that $V(x, y)$ has the symmetries of a square.[1] For our purposes, this means that

$$V(x, y) = V(y, x) = V(x, -y), \tag{2}$$

along with any further permutations you can find by repeatedly applying these kinds of identities (there are 7 distinct inequalities of the form $V(x, y) = \cdots$, if you want to find them all).

B1. Show that (2) implies that $B = 0$ and $A = C$.

B2. Conclude that the symmetries of the effective Lagrangian L are actually *larger* than merely the symmetries of a square.[2]

10 **C:** If we consider higher (fourth order) terms in V, then we can explicitly see that there are only the symmetries of the square.

C1. Write down the most general possible fourth order terms in V which are compatible with (2).

C2. What is the regime of validity where the leading order approximation in (1) is valid?

Problem 2 (Profit maximization): In standard economics, a rational firm seeks to maximize its profit over time. Suppose it sells it a single good at price $p(t)$ using a time dependent price it can control. $N(p)$ customers are willing to buy the good at price p, and is conventionally called the **demand curve** and should be a decreasing function: $\mathrm{d}N/\mathrm{d}p < 0$. The seller's profit from time $t = 0$ to time $t = T$ is given by (optimally):

$$\mathit{\Pi} = \int_0^T \mathrm{d}t \; p(t)N(p(t)). \tag{3}$$

5 **A:** How should a seller maximize $\mathit{\Pi}$ as given by (3)?[3] Deduce the profit-maximizing condition(s) for the optimal price p_* that the seller will choose.

[1] This can be precisely defined, but unfortunately this elegant mathematics is beyond the scope of this class!

[2] *Hint:* Under what rotations is your effective L invariant?

[3] *Hint:* Don't overthink this! There is no dynamics in this problem yet. Your answer should be only be a couple of lines.

Now, suppose that the seller cannot simply adjust the price and meet demand instantaneously; a seller has to build new factories to increase production, etc. To account for these microscopic details from the spirit of effective theory, we propose adjusting (3) to

$$\Pi = \int_0^T \mathrm{d}t \left[pN(p) - \frac{a}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}t}\right)^2 \right]. \tag{4}$$

where now p represents $p(t)$; here $a > 0$ is a phenomenological parameter. Now that Π depends on time, we can use variational calculus (a la the principle of least action) to analyze the seller behavior.

5 **B:** Without solving any equations, give a sketch of what you think the optimal seller would choose $p(t)$ to be, assuming $p(0) \neq p_*$. You don't need to label or compute the scaling of any time scales yet.

10 **C:** Now, as a good physicist would, you might also try to make an analogy between the seller dynamics and the motion of a one-dimensional particle in mechanics.

C1. Describe this analogy. What is the mass of the particle? What is the potential energy?

C2. What does the "typical" trajectory look like in this mechanical analogy?

C3. Compare to your answer from part **B** in a sentence or two.

Evidently, many mechanical trajectories are quite different from the seller's optimal behavior; you should be able to see by eye that most of these mechanically-inspired trajectories $p(t)$ are far from profit maximizing ones. Yet mathematically variational calculus should be valid; and our analogy also appears to have some mathematical precision to it. To understand what is happening, we will find it helpful to focus in on a toy version of our problem which captures its essence: the behavior of a seller for p close to p_*.

5 **D:** Using effective theory principles, explain in a sentence why

$$pN(p) = b - \frac{c}{2}(p - p_*)^2 + \cdots , \tag{5}$$

where $b, c > 0$ are constants.

10 **E:** Now let

$$x = C_1(p - p_*), \tag{6a}$$
$$\tilde{t} = C_2 t. \tag{6b}$$

E1. Show that by a clever choice of C_1 and C_2, you can convert the mathematical problem we are trying to understand into a slightly simpler looking one of trying to minimize a different action,

$$S[x] = \frac{1}{2}\int_0^T \mathrm{d}t \left[x^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \right]. \tag{7}$$

where here and below, for simplicity, we will just neglect to write a tilde on t and T.

E2. What is the two-parameter family of trajectories which extremize $S[x]$?

15 **F:** Now we can carry out a very explicit analysis of the minimization problem faced by the seller.

F1. Show explicitly (by going through a few steps of the derivation of the Euler-Lagrange equations) that these trajectories $x(t)$ are only extrema of $S[x]$ when we hold fixed both $x(0)$ and $x(T)$.

F2. Further explain why these trajectories *minimize* S subject to these boundary conditions on $x(t)$.

F3. We should assume that our profit-maximizing seller picks an optimal strategy by optimizing over all $x(T)$ in order to minimize $S[x]$. Do so, and thus deduce the optimal strategy $x(t)$.

F4. Compare to your answers from parts **B** and **C**; does this resolve all earlier puzzles?

20 **Problem 3:** Sometimes in physical problems, the principle of least action does not lead to a unique solution when we specify the initial and final points on a trajectory. Consider the action

$$S[x] = \int_0^T \mathrm{d}t \, \left[\frac{1}{2}m\dot{x}^2 - V(x)\right] \tag{8}$$

for a particle moving on the line $-\infty < x(t) < \infty$.[4]

1. Choose a time T, and initial/final positions $x(0)/x(T)$, such that there are at least 2 extremal trajectories of $S[x]$ beginning and ending at the specified points.
2. Choose a time T, and initial/final positions $x(0)/x(T)$, such that there are *no* extremal trajectories.

15 **Problem 4 (Parity-breaking dynamics):** Let $x_i = (x_1, \ldots, x_d)$ denote the coordinates of the particle in the d-dimensional plane. Suppose that its motion is governed by an action

$$S[x] = \int \mathrm{d}t \left[\frac{1}{2}m\dot{x}_i\dot{x}_i - V(x) + L'\right], \tag{9}$$

where L' will correspond to perturbations which break parity, but otherwise preserve the rotational symmetry of space. To break parity, we are allowed to write down terms involving the fully antisymmetric Levi-Civita tensor $\epsilon_{i_1\cdots i_d}$, so long as all of its indices are contracted with x_i, $\dot{x}_i$, etc. To avoid writing down terms that we could consider as part of $V(x)$, demand that L' must include at least one time derivative.

Note that in (9), and more generally in this class, when we have a term with repeated indices ($\dot{x}_i\dot{x}_i$), a sum over the repeated index is implied (this is Einstein's summation convention).

1. Write down the physical terms in L' (i.e., those which change the equations of motion) with the fewest numbers of x and t-derivatives which break parity for $d = 1, 2, 3, 4, 5$. Explain how the pattern will continue to all higher d.
2. Is there any dimension where the parity-breaking term will dominate over one of the standard terms (m or V) written in (9)? If so, which one(s)? Explain your answer.
3. In which dimensions does the leading-order parity breaking term also break time-reversal symmetry: i.e. $L' \to -L'$ if we send $t \to -t$ (thus this flips the sign of each derivative)?

[4] *Hint:* You can use the harmonic oscillator potential for this problem.