Homework 10

Due: October 31 at 11:59 PM. Submit on Canvas.

Problem 1 (Gauge transformations): Consider a Lagrangian L for a charged particle moving in three spatial dimensions:

$$L = \frac{1}{2}m\dot{x}_i\dot{x}_i - \Phi + A_i\dot{x}_i, \tag{1}$$

where we assume that Φ and A_i do not depend on time t, and set the charge of the particle to be 1.

A: Consider the following gauge transformation:

$$A_i \to A_i + \frac{\partial\lambda}{\partial x_i}. \tag{2}$$

Here λ is an arbitrary function.

- A1. Give a *short* calculation and/or explanation that explains why this gauge transformation cannot affect the equations of motion, within the context of Lagrangian mechanics.
- A2. Now consider the Hamiltonian formulation of this problem (cf Lecture 23). Show that under the gauge transformation we can alternatively think that A_i was unchanged, while

$$p_i \to P_i = p_i + \frac{\partial\lambda}{\partial x_i}. \tag{3}$$

B: Independently of electromagnetism, we can always consider a transformation of the phase space coordinates of the form (3). Namely, given a now arbitrary Hamiltonian $H(x^i, p^i)$, with standard Poisson bracket $[x^i, p^j] = \delta^{ij}$, we can consider the system in coordinates

$$X^i = x^i, \tag{4a}$$

$$P^i = p^i + \frac{\partial\lambda}{\partial x^i}. \tag{4b}$$

- B1. Show that (4) is a type 2 canonical transformation.
- B2. What is the Hamiltonian $H(X^i, P^i)$? Don't plug in for results from **A**, and keep $H(x^i, p^i)$ general.
- B3. As the transformation is canonical, it should leave the equations of motion invariant. Show explicitly that this works: namely, how equations for $\dot{X}^i$ and $\dot{P}^i$ reduce to those for $\dot{x}^i$ and $\dot{p}^i$.

Problem 2 (Rigid body rotation): We revisit the problem of rigid body rotation using the Poisson bracket formulation of mechanics. An interesting perspective is that the "degrees of freedom" of the problem are the angular momentum components L_1, L_2, L_3 in the body frame. Their Poisson brackets are (note the relative sign difference from Lecture 24, because the body frame co-rotates with the object!):

$$[L_1, L_2] = -L_3, \tag{5a}$$

$$[L_2, L_3] = -L_1, \tag{5b}$$

$$[L_3, L_1] = -L_2. \tag{5c}$$

IThe Hamiltonian for rigid body rotation is

$$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}. \tag{6}$$

20 **A:** Use the Poisson bracket "equation of motion" to find first order equations for $\dot{L}_1$, $\dot{L}_2$, $\dot{L}_3$. Show that your answer agrees with what we found in Lectures 9 and 10.

15 **B:** We are seemingly able to apply Hamiltonian mechanics to the problem, and yet it also seems that the dynamics takes place on an *odd-dimensional phase space*? This cannot be possible – all symplectic manifolds are *even-dimensional*, as we discussed in Lecture 25. The resolution to this puzzle is that the symplectic manifold of interest here is actually the 2-sphere S^2!

B1. To see how this is possible, show one of the following Poisson brackets (commutators) vanishes (the other two calculations are basically identical), using only commutator identities and (5):

$$0 = \left[L^2, L_1\right] = \left[L^2, L_2\right] = \left[L^2, L_3\right]. \tag{7}$$

for $L^2 = L_1^2 + L_2^2 + L_3^2$. (Again, note the analogy to the quantum angular momentum algebra!)

B2. In one or two sentences, state why we can restrict the dynamics to a subspace of fixed L^2, and therefore that the phase space for the problem is S^2.

B3. To verify that this is indeed a symplectic manifold, we need to find the symplectic form ω_{IJ} as a 2×2 matrix. To do this in some local coordinates, let us fix for simplicity $L^2 = 1$, and then parameterize

$$L_1 = \sin\theta\cos\phi, \tag{8a}$$
$$L_2 = \sin\theta\sin\phi, \tag{8b}$$
$$L_3 = \cos\theta. \tag{8c}$$

Since ω is antisymmetric, the only non-vanishing components will be $\omega_{\theta\phi} = -\omega_{\phi\theta}$. Show that if

$$\omega_{\theta\phi} = \sin\theta, \tag{9}$$

then the Poisson brackets of the angular momenta agree with (5).[1]

Remarkably, we have thus found that the 2-sphere S^2 is also a symplectic manifold, and a valid phase space for Hamiltonian mechanics, even though it is compact!

Problem 3 (Hidden symmetry of the harmonic oscillator): Consider the n-dimensional isotropic harmonic oscillator, whose classical (or quantum) Hamiltonian is given by:

$$H = \frac{p_i p_i + q_i q_i}{2}. \tag{10}$$

Here we are using dimensionless units; the index $i = 1, \ldots, n$. Certainly this problem is invariant under rotations (in n-dimensions), but Hamiltonian mechanics reveals a much larger symmetry group!

As in Lecture 24, collect ps and qs into ξ^I. We can write

$$H = \frac{1}{2}\delta_{IJ}\xi^I\xi^J, \tag{11}$$

where δ_{IJ} is a $2n \times 2n$ identity matrix.

[1] *Hint:* Be careful with raised and lowered IJ indices. What is ω^{IJ}, as defined in Lecture 24? What was the formal definition of the Poisson bracket?

15 **A:** First, we explicitly hunt for some infinitesimal symmetries of this system by clever "inspection". Let us focus on symmetries of the form
$$\xi^I \to \xi^I + \epsilon^I{}_J \xi^J, \tag{12}$$
where $\epsilon^I{}_J$ is infinitesimally small.

A1. The first property we desire of a symmetry is that it is a coordinate transformation that leaves H invariant. Explain why this leads to[2]
$$\epsilon^K{}_I \delta_{KJ} + \delta_{IK} \epsilon^K{}_J = 0. \tag{13}$$

A2. We would also like for our infinitesimal symmetries to be canonical transformations. Explain why this means that
$$\epsilon^K{}_I \omega_{KJ} + \omega_{IK} \epsilon^K{}_J = 0. \tag{14}$$
Here and below, assume the "canonical" choice for ω_{IJ} given at the beginning of Lecture 24.

A3. Show that the most general such transformation is
$$\epsilon^I{}_J = \begin{pmatrix} -B & -A \\ A & -B \end{pmatrix} \tag{15}$$
where A is a symmetric $n \times n$ matrix and B is an antisymmetric $n \times n$ matrix. As in Lecture 24, the first block row is for q_i, and the second for p_i. In index notation we can write:
$$q_i \to q_i - A_{ij} p_j - B_{ij} q_j, \tag{16a}$$
$$p_i \to p_i + A_{ij} q_j - B_{ij} p_j. \tag{16b}$$
where $A_{ij} = A_{ji}$ and $B_{ji} = -B_{ij}$.

15 **B:** As we saw in Lecture 27, in Hamiltonian mechanics we could also start by finding the conserved quantities and then deduce the most general symmetry transformations. Now, we proceed with this perspective.

B1. First, let us obtain a useful mathematical result. Show that
$$\left[M_{IJ}\xi^I\xi^J, N_{KL}\xi^K\xi^L\right] = 2\left(M_{JI}\omega^{IK}N_{KL} + M_{LI}\omega^{IK}N_{KJ}\right)\xi^J\xi^L. \tag{17}$$
Here $M_{IJ} = M_{JI}$ and $N_{KL} = N_{LK}$ are symmetric matrices. Since the matrix in parentheses above is symmetric, this means that we have a (closed) Lie algebra corresponding to the Poisson brackets of all quadratic polynomials in ξ!

B2. Show that all quadratics $M_{IJ}\xi^I\xi^J$ that have vanishing Poisson bracket with H – which is itself quadratic: (11) – are of the form
$$M_{IJ} = \frac{1}{2}\begin{pmatrix} -A & B \\ -B & -A \end{pmatrix}. \tag{18}$$
where A is a symmetric and B is an antisymmetric $n \times n$ matrix. Denote any quadratic of the form (18) as F_M for short, moving forward.

B3. Show that F_M generates the infinitesimal canonical transformation (16).

[2] *Hint:* This is the $2n$-dimensional generalization of what we did in Lecture 8.

B4. Will Poisson brackets $[F_{M_1}, F_{M_2}]$ form a closed algebra? Namely, will $[F_{M_1}, F_{M_2}] = F_{M_3}$ for some other polynomial constrained by (18)?[3]

5 **C:** Show that the Lie algebra of Poisson brackets $[F_{M_1}, F_{M_2}]$ is the same as (isomorphic to) the Lie algebra $\mathfrak{u}(n)$, which is the commutator algebra of the generators of $n \times n$ unitary matrices. Namely, show how you can think of $M_{1,2}$ as generators of unitary transformations on $\mathbb{C}^n$, and relate the Poisson bracket $[F_{M_1}, F_{M_2}]$ to a *matrix commutator* of complex matrices, $[M_1, M_2]$.[4]

This hidden $\mathrm{U}(n)$ symmetry of the isotropic oscillator is responsible for the extreme degeneracy of the quantum mechanical oscillator – it is similar to the origin of the degeneracy of the nonrelativistic hydrogen atom (cf Homework 9), albeit arising from a different "hidden" symmetry group.

Problem 4 (Parametric resonance): Consider the following Hamiltonian system, corresponding to a driven oscillator:

$$H = \frac{p^2}{2} + [1 + \epsilon(t)]\,\frac{q^2}{2}. \tag{19}$$

We can think of this as a crude model for a child swinging on a swing set and trying to move their body in such a way as to begin to swing. In "natural units" the fundamental frequency of the oscillator is 1, while the child's motion (which we assume does not change very much the fundamental frequency of the swing set at any given time) is periodic: $\epsilon(t + \tau) = \epsilon(t)$, with $|\epsilon(t)| < 1$.

10 **A:** We focus on the dynamics at times $t = 0, \tau, 2\tau, \ldots$.

A1. Explain why, given $q(0)$ and $p(0)$, there is a 2×2 real matrix S such that

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = \mathsf{S} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}. \tag{20}$$

A2. Conclude that $\det(\mathsf{S}) = 1$.[5]

A3. By considering the most general possible form of the eigenvalues of S, conclude that the child's motion will lead to an instability (and thus large amplitude of swinging) whenever $|\mathrm{tr}(\mathsf{S})| > 2$. This phenomenon is called **parametric resonance**.

10 **B:** Consider the choice

$$\epsilon(t) = \begin{cases} \epsilon & 0 \le t < \frac{1}{2}\tau \\ -\epsilon & \frac{1}{2}\tau \le t < \tau \end{cases}. \tag{21}$$

with $\epsilon(t + \tau) = \epsilon(t)$ used to define ϵ for times $t \ge \tau$. Show that for certain values of τ and ϵ, the criterion of A3 is satisfied, and thus there is an instability to large amplitude oscillation. Sketch in the (τ, ϵ) plane the location of the instability, and comment on the result. You can take $\epsilon \ll 1$.

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[3] *Hint:* Very little calculation is needed!

[4] *Hint:* Recall from quantum mechanics that the generators of a unitary transformation are $\mathrm{i}T$, where $T = T^\dagger$. Then, the most clever way to see how a complex $n \times n$ matrix emerges is to try and package (16) using complex numbers.

[5] *Hint:* Time-translation is a canonical transformation. This gives you a constraint on S.