Homework 11

Due: November 7 at 11:59 PM. Submit on Canvas.

20 **Problem 1 (WKB approximation):** The Hamilton-Jacobi equation is closely related to the semiclassical WKB approximation from quantum mechanics. Consider the Schrödinger equation in d-dimensions:

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x_i \partial x_i} + V(x_i)\Psi = \mathrm{i}\hbar \frac{\partial \Psi}{\partial t}, \tag{1}$$

and consider the ansatz (don't worry that it isn't normalized)

$$\Psi(x_i, t) = \rho(x_i, t)\mathrm{e}^{\mathrm{i}S(x_i,t)/\hbar}. \tag{2}$$

Plug in (2) into (1). Show that at leading order in the $\hbar \to 0$ limit, you recover the Hamilton-Jacobi equation for S, for a physically sensible Hamiltonian.

Problem 2 (Meson): A toy model for a meson (elementary particle made up of a quark and anti-quark) is a one-dimensional quantum system with Hamiltonian

$$H = c|p| + F|x|. \tag{3}$$

In this crude model, you can think of the meson as made up of a stationary quark and an anti-quark, which is effectively massless (and thus has ultrarelativistic kinetic energy). There is a "flux tube" which acts on the anti-quark with a constant tension of strength F that pulls it towards the quark; the Hamiltonian above is the sum of kinetic and potential energies.

20 **A:** The phase space for this problem, as written above, is formed by canonical coordinate pair (x, p).

- A1. Sketch the surface of constant $H = E$ in phase space. Sketch (without solving any equations) how the system will flow through phase space under Hamiltonian evolution.
- A2. Describe the solution to Hamilton's equations and find agreement with your qualitative sketch. While you do not need to write down a solution to the equations for all time, do give an explicit formula for the period T of the oscillations.

20 **B:** Now, let us re-solve this problem in action-angle coordinates.

- B1. Evaluate the "action" coordinate $J = (2\pi)^{-1} \oint \mathrm{d}x\, p$.
- B2. What is the Hamiltonian $H(J)$?
- B3. What is the period T of the oscillations? Confirm agreement with A2.

Problem 3 (Parabolic coordinates): Consider the 2d Kepler problem with Lagrangian (in rectangular coordinates)

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{k}{\sqrt{x^2 + y^2}}. \tag{4}$$

10 **A:** In this problem, we are going to work with parabolic coordinates (a, b):

$$x = \frac{a-b}{2}, \tag{5a}$$
$$y = \sqrt{ab}. \tag{5b}$$

A1. Determine $\dot{x}$ and $\dot{y}$ in terms of $a, b, \dot{a}, \dot{b}$, using the chain rule.

A2. Determine L in terms of $a, b, \dot{a}, \dot{b}$.

10 **B:** Now, do the Legendre transform to a Hamiltonian. Show that

$$H = \frac{2\left(ap_a^2 + bp_b^2\right)}{m(a+b)} + \frac{2k}{a+b}. \tag{6}$$

20 **C:** Now, let us use the Hamilton-Jacobi method.

C1. Write down the Hamilton-Jacobi equation for $S(a, b, t)$.

C2. Plug in the guess $S = W_a(a) + W_b(b) - Et$, and show that the Hamilton-Jacobi equation can be solved by separation of variables. Solve the equation by determining W_a and W_b in terms of integrals (which you do not need to explicitly evaluate). Expect to introduce a constant of integration at some point.

15 **Problem 4 (Magnetic lensing):** Consider the Hamiltonian (in cylindrical coordinates)

$$H = \frac{p_r^2}{2m} + \frac{(p_\theta - qr^2h(z))^2}{2mr^2} + \frac{p_z^2}{2m}. \tag{7}$$

This Hamiltonian describes the motion of a charged particle of charge q and mass m in a particular kind of magnetic field. This magnetic field leads to "lensing", wherein a source of particles emitted at $(r, z) = (0, -z_1)$, with $z_1 > 0$, can be re-focused at $(r, z) = (0, z_2)$ with $z_2 > 0$.

Assume that the function $h(z)$ vanishes for $|z| > a$. You can also assume that $a \ll z_1$.

1. Argue that it is reasonable to neglect θ and p_θ coordinates in this problem, given the source of particles of interest.

2. For the remaining r and z coordinates, write down the Hamilton-Jacobi equation for $S = W(r, z) - Et$.

3. Although you cannot solve the resulting equation by separation of variables, you can expand out

$$W(r, z) = W_0(z) + W_2(z)r^2 + \cdots. \tag{8}$$

Find the solution for $W_0(z)$, and a differential equation for $W_2(z)$.

4. Use the fact that $h(z)$ vanishes for $z < -a$ and $z > a$ to deduce the form of $W_{0,2}(z)$ in each region. You can assume for simplicity that the length scale a is very small, while h^2a remains finite, if this helps you to simplify the calculation at all.

5. Use continuity at $z = \pm a$ to stitch together the form of $W_{0,2}(z)$ in the entire domain.

6. Use your solution to confirm that the magnetic lens works. Find the value of z_2 at which the particles are re-focused.