PHYS 5210: Graduate Classical Mechanics **Fall 2022**

Homework 12

Due: November 14 at 11:59 PM. Submit on Canvas.

25 **Problem 1 (Adiabatic compression of a gas):** Consider a molecule in a one-dimensional non-relativistic gas. The molecule has mass m and bounces back and forth between the walls of a container of length L. The length L is assumed to slowly (adiabatically) vary in time. While in this problem we focus on the dynamics of a single molecule, we can use its behavior to estimate what happens for the gas as a whole.

1. Find an adiabatic invariant for this problem as a function of the molecule's energy, E.
2. Deduce, as L changes, how the temperature T of the gas (which is proportional to E), will vary.
3. The pressure P in the gas is proportional to the average force per unit time applied to the left (e.g.) edge of the box, by the molecule bouncing off of it. How will P vary as a function of L?

Problem 2: Consider the two-dimensional (coupled) harmonic oscillator

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{A}{2}\left(x^2 + y^2\right) + Bxy. \tag{1}$$

15 **A:** This problem can be exactly solved within Hamiltonian mechanics, as follows.

A1. Show that the following transformation is canonical:

$$\begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix} \tag{2}$$

A2. What value(s) of θ should you choose in order to simplify the solution of the coupled oscillator problem?[1] After making this choice, write down $H(x_1, x_2, p_1, p_2)$.

15 **B:** Find action-angle variables $(J_1, J_2, \phi_1, \phi_2)$ for this problem, and deduce the Hamiltonian $H(J_1, J_2)$. As part of your solution, give an *explicit* canonical transformation from (x_1, x_2, p_1, p_2) to, or from, the action-angle variables.[2]

10 **C:** Let us now interpret our findings in the language of integrable systems, following Lecture 30.

C1. Explain why H is integrable – i.e., it formally meets the criteria of Lecture 30.

C2. Describe how phase space is written in terms of invariant tori on top of a two-dimensional manifold with coordinates (J_1, J_2). A geometric picture may be hard to draw, but communicate how the Liouville-Arnold Theorem is obeyed in this system as best you can.

[1] *Hint:* One solves coupled oscillators by writing H as the sum of two independent oscillators $H_1 + H_2$.

[2] *Hint:* Use results from Lecture 31 to help you – almost no new computation necessary.

C3. For what values of A and B will motion on the invariant tori be commensurate? What about incommensurate?

5 **D:** Provide an example of a perturbation to H that is not quadratic in $x_{1,2}/p_{1,2}$, yet maintains integrability with the action-angle variables of **B**. The perturbation may not be physically reasonable, but be sure in your answer to explain why the problem is nevertheless still exactly solvable.

15 **E:** Now, consider perturbing the Hamiltonian to $H' = H_0 + \epsilon H_1$, where H_0 is the Hamiltonian given in (1), and

$$H_1 = C\left(x^4 + y^4\right) + Dx^2y^2. \tag{3}$$

E1. Write H_1 in terms of the action angle variables $(J_1, J_2, \phi_1, \phi_2)$ found in **B**.

E2. Follow Lectures 31 and 32. Describe how one can, in principle, look for new action-angle coordinates for the perturbed system. Find H in terms of the new action-angle coordinates, using perturbation theory, to first order in ϵ.

15 **F:** Let us now more explicitly carry out the canonical transformation to new action-angle coordinates $(J_1, J_2, \phi_1, \phi_2)$, to first order in ϵ, from the old action-angle coordinates, which we'll now denote as $(J_{1,0}, J_{2,0}, \phi_{1,0}, \phi_{2,0})$ First, assume that $B \neq 0$.

F1. Find the type 2 generating function from old to new action-angle variabes, to first order in ϵ.

F2. Can perturbation theory break down at first order in ϵ?

15 **G:** Now, return to the case of $B = 0$. In this case, a bit more care is needed to implement perturbation theory, analogous to the case of degenerate perturbation theory in quantum mechanics.

G1. Find action-angle variables $(J_0, J_0', \phi_0, \phi_0')$ in which the unperturbed Hamiltonian $H_0 \propto J_0$.[3]

G2. Implement a kind of "degenerate" perturbation theory, in which you first find a canonical transform to coordinates (J, J_0', ϕ, ϕ_0'), in which H is independent of ϕ to first order in ϵ. Argue that at first order, the problem becomes to a new (somewhat complicated) Hamiltonian system for a single pair of coordinates (J_0', ϕ_0').

[3] *Note:* You can do this without a lot of calculation, but you may need to think carefully about normalization factors to ensure that $\phi_0 \sim \phi_0 + 2\pi$ and $\phi_0' \sim \phi_0' + 2\pi$ are periodic angle variables.