Homework 13

Due: December 5 at 11:59 PM. Submit on Canvas.

Problem 1 (Stirred fluid): A fluid is alternately stirred by two thin rods at $x = \sigma(t)a$, where

$$\sigma(t) = \begin{cases} 1 & 0 \le t \le T, 2T \le t \le 3T, \ldots \\ -1 & T \le t \le 2T,\ 3T \le t \le 4T, \ldots \end{cases}. \tag{1}$$

The stirring creates a **vortex** flow pattern, where the local fluid velocity takes the form

$$v_x = \frac{-by}{y^2 + (x - a\sigma(t))^2}, \tag{2a}$$

$$v_y = \frac{b(x - a\sigma(t))}{y^2 + (x - a\sigma(t))^2}. \tag{2b}$$

Take $a, b > 0$ to be time-independent and constant.

A: It makes sense to consider a tracer particle (such as a drop of dye) moving in this fluid according to the equations

$$\dot{x} = v_x(x, y, t), \tag{3a}$$

$$\dot{y} = v_y(x, y, t). \tag{3b}$$

Show that these equations actually follow from Hamilton's equations, with

$$H(x, y, t) = -\frac{b}{2} \log\left((x - a\sigma(t))^2 + y^2\right), \tag{4}$$

and Poisson bracket $[x, y] = 1$ (meaning x, y are canonically conjugate).[1]

B: Let's find the solution to these equations. Let $(x_n, y_n) = (x(nT), y(nT))$.

B1. Rescale $x_n \to a\tilde{x}_n$, $y_n \to a\tilde{y}_n$, and $t \to T\tilde{t}$. Deduce that there is one interesting dimensionless parameter in the problem:

$$q = \frac{bT}{a^2}. \tag{5}$$

B2. Given initial conditions (x_0, y_0), explain why for $0 \le t \le T$ the particle will move on a circular trajectory.[2] Determine (x_1, y_1).

B3. Let $\sigma_{0,2,4,\ldots} = 1$ and $\sigma_{1,3,5,\ldots} = -1$. Continue the logic of B2 to show that

$$\tilde{x}_{n+1} = \sigma_n + (\tilde{x}_n - \sigma_n) \cos \frac{q}{(\tilde{x}_n - \sigma_n)^2 + \tilde{y}_n^2} - \tilde{y}_n \sin \frac{q}{(\tilde{x}_n - \sigma_n)^2 + \tilde{y}_n^2}, \tag{6a}$$

$$\tilde{y}_{n+1} = (\tilde{x}_n - \sigma_n) \sin \frac{q}{(\tilde{x}_n - \sigma_n)^2 + \tilde{y}_n^2} + \tilde{y}_n \cos \frac{q}{(\tilde{x}_n - \sigma_n)^2 + \tilde{y}_n^2}. \tag{6b}$$

[1] *Hint:* Recall that $[x, H] = [x, x]\partial_x H + [x, y]\partial_y H$, etc.

[2] *Hint:* Can you find any conservation laws for $0 \le t \le T$? Try to avoid integrating any differential equations directly; instead show that the angular velocity on the circle is a constant.

20 **C:** Starting, e.g., with the `Mathematica` notebook for Lecture 35, write a short code to iterate (6) for many steps n, and various initial conditions, drawn from $|\tilde{x}_0|, |\tilde{y}_0| \lesssim 1$. Argue convincingly for a transition from integrability to chaos, with their coexistence at intermediate q.

We see that relatively "predictable" ways of stirring a fluid can lead to rapid (chaotic) mixing!

25 **Problem 2:** Let S be the subset of the real line between 0 and 1, consisting of numbers x whose decimal expansions (in base 10, as above) contain only even digits: 0, 2, 4, 6, 8. What is the box dimension of S?

Problem 3 (Decimal shift map): A rare example of a chaotic one-dimensional map which is exactly solvable is the "decimal shift map", defined as follows. Let us define (for $x \in \mathbb{R}$)

$$\{x\} = x - \lfloor x \rfloor, \tag{7}$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$. More transparently, $\{x\}$ is the "decimal part of x" – for example, $\{3.54\} = 0.54$, and in general $0 \leq \{x\} < 1$. The **decimal shift map** (DS) is then

$$x_{n+1} = \{10x_n\}. \tag{8}$$

10 **A:** Let us begin by looking for fixed points and cycles of DS.

- A1. Show that there are 9 fixed points x_* obeying $x_* = \{10x_*\}$. What are they?
- A2. How many period-2 cycles are there? What are they?

10 **B:** We can define the Lyapunov exponent λ for DS as follows: given two initial conditions x_0 and x_0' obeying $|x_0 - x_0'| < \epsilon$, for sufficiently small ϵ, then

$$\lambda \approx \frac{1}{n} \log \frac{|x_n - x_n'|}{|x_0 - x_0'|}. \tag{9}$$

In this expression, we think of first taking $\epsilon \to 0$, and then $n \to \infty$. Calculate λ analytically for DS.

Problem 4: Consider the logistic map of Lecture 36.

10 **A:** Set $r = 3.83$. You can use (with suitable edits) the provided `Mathematica` notebook.

- A1. Show that typical initial conditions flow towards a stable period-3 cycle.
- A2. Use a numerical linear stability analysis to demonstrate the stability of this period-3 cycle.

10 **B:** Let $S \subset [0, 1]$ be the subset of initial conditions x_0, such that as $n \to \infty$, $x_{3n} \approx 0.504$.

- B1. Numerically investigate the geometry of S. Argue that it has very intricate fractal structure, but nevertheless should have box dimension 1. This is called a "fat fractal".
- B2. Numerically determine the box dimension d of ∂S, the *boundary* of the set S, following Lecture 39, by simply counting the number of "boxes" of a given size containing points in ∂S[3]. You should find this procedure behaves pretty poorly in practice.

10 **C:** There is a more accurate algorithm to efficiently estimate d numerically. Run the logistic map for two initial conditions x_0 and $x_0 + \delta$, for various choices of x_0. Let $f(\delta)$ denote the fraction of simulations where x_{3n} is very different for the two simulations.[4]

- C1. Argue that $f(\delta) \sim \delta^{1-d}$, and explain why this algorithm is better.
- C2. Numerically implement and run this algorithm. Estimate d with error $\lesssim 0.01$.

[3] *Note:* If set $R = (0.3, 0.5] \cup [0.6, 0.7]$, then $\partial R = \{0.3, 0.5, 0.6, 0.7\}$.

[4] Strictly speaking, to check whether they land in S, we also need to restrict to only the initial conditions where $x_{3n} \approx 0.504$, but you can ignore this for simplicity now – all 3 "basins" behave similarly.