PHYS 5210: Graduate Classical Mechanics **Fall 2022**

Homework 2

Due: September 7 at 11:59 PM. Submit on Canvas.

Problem 1 (Polynomial boost symmetry): Consider the motion of a free particle along a line, with trajectory $x(t)$. Suppose that the theory is invariant under a polynomial "boost" in the trajectory $x(t)$:

$$x(t) \to x(t) + \epsilon_0 + \epsilon_1 t + \cdots + \epsilon_n t^n \tag{1}$$

for some integer $n \geq 1$, and infinitesimal parameters $\epsilon_0, \ldots, \epsilon_n$.

20 **A:** In our universe, we (in the non-relativistic limit) have **Galilean invariance**, where $n = 1$.

- A1. What is the physical interpretation of the two parameters ϵ_0 and ϵ_1? A few words suffice.
- A2. Show that (when particle endpoints are fixed), the action

$$S = \frac{m}{2} \int \mathrm{d}t \; \dot{x}^2 \tag{2}$$

  is the lowest derivative action describing a theory invariant under Galilean symmetry *to first order* in the ϵ parameters, and thus the one we would write down using effective theory.
- A3. Although S is not invariant at quadratic order in ϵ, argue this is not a problem.
- A4. What is the next most relevant (fewest derivatives) term that should be included in L?

10 **B:** Now, let us consider the case where $n = 2$. Generalize the arguments above and show that the minimal Lagrangian within effective theory is

$$S = \frac{a}{2} \int \mathrm{d}t \; \ddot{x}^2. \tag{3}$$

15 **C:** The theory above is an example of a system where we need to consider higher derivative terms in the Lagrangian.

- C1. Generalize the derivation of the Euler-Lagrange equations from Lecture 1 to a system with a higher derivative Lagrangian $L(x, \dot{x}, \ddot{x})$. As part of your derivation, you should deduce the "right" boundary conditions to impose on $x(t)$ (or its derivatives) for the principle of least action to lead to a simple Euler-Lagrange equation.
- C2. What is the equation of motion for x given the action (3)? What is its general solution?

15 **D:** Generalize the derivation of Noether's Theorem (being careful about higher derivative terms!) to find the three conserved quantities of the theory (3), associated with the parameters $\epsilon_{0,1,2}$.

20 **Problem 2 (The geodesic equation):** In general relativity, one of the most important concepts is the notion of a geodesic in curved spacetime – the "minimal length" path between two points. Falling in a "gravitational field" becomes understood as simply traversing the path of minimal proper time in curved spacetime. The action of a relativistic particle in curved space is given by

$$S = -mc \int \mathrm{d}\theta \sqrt{-g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\theta} \frac{\mathrm{d}x^\nu}{\mathrm{d}\theta}}, \tag{4}$$

where the matrix $g_{\mu\nu}(x)$, called the **metric**, encodes the curvature of space. Do not make any assumptions on the form of $g_{\mu\nu}$, other than that it is symmetric and invertible. In flat space, $g_{\mu\nu}(x) = \eta_{\mu\nu}$. The inverse metric is denoted as $g^{\mu\nu}$ and obeys

$$g^{\mu\nu} g_{\nu\lambda} = \delta^\mu{}_\lambda. \tag{5}$$

1. Given that the argument of the square root in (4) is always strictly positive, argue that we can always choose θ to be the proper time τ, which is defined so that $-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\theta} \frac{\mathrm{d}x^\nu}{\mathrm{d}\theta} = c^2$ is a constant.

2. Using this trick to simplify expressions in the Euler-Lagrange equations, *after* evaluating $\delta S / \delta x^\mu$, show that the equations of motion for this action are

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0, \tag{6}$$

   where

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho} \right). \tag{7}$$

   This is called the **geodesic equation**, and describes how massive particles move in curved space.

Problem 3: The relativistic particle action that we discussed in Lectures 3 and 4 admits a very large family of symmetries called the Lorentz group (at least when $A_\mu = 0$). The continuous symmetry transformations that we can use in Noether's Theorem to deduce new conservation laws were discussed in Lecture 3.

10 **A:** We begin by discussing the full symmetries of the free particle ($A_\mu = 0$).

- A1. Using the θ-parameterized action (which is more manifestly invariant under relativistic symmetries), deduce the conserved quantities arising translations and Lorentz transformations.
- A2. Why is there no interesting conserved quantity associated with θ-translation symmetry?
- A3. Set $\theta = t$ and discuss the resulting conserved quantities – are they what you expect?

10 **B:** Now consider putting a relativistic particle of charge q in a magnetic field of strength B, oriented in the z-direction. A suitable vector potential that does this is

$$(A_t, A_x, A_y, A_z) = \left(0, -\frac{By}{2}, \frac{Bx}{2}, 0\right). \tag{8}$$

- B1. Which of the above symmetries is L invariant under? What about the theory (i.e. equations of motion)?
- B2. Use the conserved quantities found in part **A** to strongly (completely) constrain the form of particle trajectories.

Problem 4 (Scale invariance): Consider a particle moving in one dimension, with trajectory $x(t)$. For simplicity if you wish, you may restrict to motion with $x > 0$. Suppose that we postulate that this particle's universe has time-translation and time-reversal symmetry, along with the invariance of the action under the combined transformations

$$x \to \lambda x, \tag{9a}$$
$$t \to \lambda^a t. \tag{9b}$$

for some constant $a > 0$, and any constant $\lambda \neq 0$. This is called **scale invariance**. In particular, we demand that the *action* S is invariant under this scaling transformation.

10 **A:** Using the principles of effective theory, let us deduce the most generic action that we can write down, assuming that the equations of motion are local in time.

- A1. Show that the most generic possible expression for the Lagrangian L that is non-trivial, consistent with the necessary symmetries, and involves time derivatives of at most second order, is[1]

$$L = \frac{m}{2} x^{a-2} \dot{x}^2 - \frac{k}{x^a}. \tag{10}$$

  where m and k are constants.
- A2. Argue that there is a canonical choice $a = 2$. Namely, for any other value of a, we can find some transformation of (x, t) that (when applied to L) gives us back the theory $a = 2$, meaning that it suffices to consider this special case.
- A3. When $a = 2$, what is the most important correction that you can write in the Lagrangian (i.e. that involves the fewest extra time derivatives)?

5 **B:** Now, consider your effective action with $a = 2$ from above.

- B1. Use Noether's Theorem to deduce two constants of motion.
- B2. One of them has a natural interpretation – what is it?
- B3. Find the general physical solution to the equations of motion for $x(t)$.

5 **C:** Take $m, k > 0$, and consider the particle incident from $x \to \infty$ (as $t \to -\infty$) with a speed of v. Does the time that it takes for the particle to bounce back depend on k?

Theories with scale invariance are quite important in modern physics and often arise as the emergent phenomenological description of some complex many-body system, especially near a phase transition. A huge part of theoretical physics research over the past 60 years has focused on understanding the implications of scale invariance in quantum field theory. In this particular problem, the scale invariance that arose was special and often goes by the name of (a part of) non-relativistic conformal symmetry.

[1] Simply checking that L is invariant (but not explaining why no other choice is) will not receive full credit.