Homework 3

Due: September 12 at 11:59 PM. Submit on Canvas.

Problem 1: In Lecture 6, we described motion of a hanging pendulum (e.g. stuck to a wall) as a physically realizable system whose configuration space is the circle S^1.

20 **A:** Suppose that we are told that $\theta = 0$ is the unique stable equilibrium for the pendulum. We would like to write down an effective theory for the pendulum's motion.

- A1. Why we are only allowed to write down $\cos\theta$ or $\sin\theta$ (or powers thereof)[1] in the Lagrangian?
- A2. It is reasonable to assert that within the effective theory for near-equilibrium dynamics, we should treat each higher power of $\cos\theta$ or $\sin\theta$ as a subleading term. Can you think of (one) reason why?
- A3. Write down $L(\theta, \dot{\theta})$ which contains the lowest order non-trivial terms in both θ and $\dot{\theta}$ which are allowed to exist on this configuration space, given that $\theta = 0$ is stable.
- A4. Compare your answer to the textbook $L = T - V$ – do you find agreement (upon relating constant coefficients)?

10 **B:** We could also try to do effective theory on this problem by doing a Taylor expansion near $\theta = 0$, following Lecture 2.

- B1. Show that upon Taylor expanding L from part **A** you find agreement with the harmonic oscillator Lagrangian predicted by effective theory in Lecture 2. (You can simply quote the oscillator's Lagrangian without derivation.)
- B2. Is the $L(\theta, \dot{\theta})$ from part **A** the unique Lagrangian that would have led to agreement with the harmonic oscillator Lagrangian in the $\theta \to 0$ limit? If not, was the effective theory approach from before incorrect?
- B3. Do you think the mathematical configuration space changed after we perform the Taylor expansion and then restrict to the $\theta \to 0$ limit?[2]

10 **C:** Mathematicians say that a Lagrangian, depending only on first derivatives, is defined on a space called the tangent bundle $\mathbb{T}\mathrm{S}^1$. $\mathbb{T}\mathrm{S}^1$ can be thought of as a cylinder ($\mathrm{S}^1 \times \mathbb{R}$), where $\theta \in \mathrm{S}^1$ and $\dot{\theta} \in \mathbb{R}$. Mathematicians say that it is impossible for $\mathbb{T}\mathrm{S}^1$ to be a compact manifold, such as the two-dimensional torus $\mathrm{S}^1 \times \mathrm{S}^1$.

Argue from a physics perspective that this makes sense – we should indeed never think of $L(\theta, \dot{\theta})$ as a map from a compact manifold to $\mathbb{R}$.

When we discuss Hamiltonian mechanics, it will turn out that this formalism *does* make sense on certain compact manifolds. We will (almost certainly) even study examples in this class! The inability to naturally study dynamics on compact *phase spaces* is a shortcoming in the Lagrangian formulation of mechanics.

[1] Using trig identities $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ can be related to powers of cos and sin, e.g.

[2] You can make an argument for either yes or no here. Whichever one you pick, however, make a convincing point. Or better yet, try to give a perspective for both yes and no.

Problem 2 (Nematic liquid crystal): A nematic liquid crystal is a phase of matter made out of rod-shaped molecules. The molecules themselves form a liquid (no translational ordering, as in a solid), yet the molecules themselves all orient their "rods" in the same direction. Interesting physics in liquid crystals arises because the manifold of all possible orientations of this rod (even for one liquid crystal molecule) is non-trivial.

In this problem, our goal is to understand using effective theory the natural kinetic energy describing the rotation of a single liquid crystal molecule. Mathematically, this amounts to finding the most "natural" Lagrangian on the manifold corresponding to the space of all possible liquid crystal configurations. This manifold is called $\mathbb{RP}^2$ (two-dimensional real projective space), and corresponds to the set of all lines in three-dimensional space $\mathbb{R}^3$ that pass through the origin: see Figure 1.

20 **A:** Alice suggests that one way to think about a point $p \in \mathbb{RP}^2$ is to draw a unit sphere of radius 1 around the origin: $x^2 + y^2 + z^2 = 1$. Let (θ, ϕ) denote the angular coordinates of the point where line p intersects the unit sphere: see Figure 1. Alice suggests that we write a Lagrangian for the liquid crystal dynamics in terms of (θ, ϕ).

- A1. What is the most sensible Lagrangian Alice should write down?
- A2. Is the identification of p with (θ, ϕ) unique? If not, do you think fixing this "problem" would change the answer to A1?

5 **B:** Bob suggests an alternative coordinate system for $\mathbb{RP}^2$, where he considers a point $(x, y, 1)$ along line p where $z = 1$ (assuming it exists). See Figure 1. Namely, he now considers the intersection of p with a plane, rather than a sphere. He then suggests the most natural Lagrangian is

$$L = A\left(\dot{x}^2 + \dot{y}^2\right) \tag{1}$$

for some constant A.

While formally speaking this L does describe a reasonable physical system, explain why it won't be the correct description of the liquid crystal's rotational dynamics. Explain in one or two sentences why Alice's approach is better.

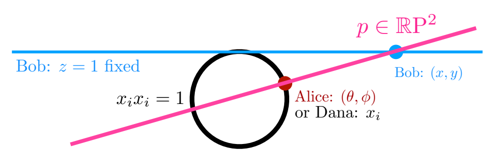

Figure 1: A sketch of a point $p \in \mathbb{RP}^2$ being a line passing through the origin, along with Alice, Bob and Dana's perspectives on how to find a more tractable "representation" of p, out of which they can each write down the Lagrangian L.

20 **C:** Charlie asserts that Bob's coordinate system is just as valid as Alice's, and it must be possible to describe correctly the dynamics of the liquid crystal using Bob's coordinate system. What Charlie explains to Bob is that he should have instead started by building a Lagrangian for a particle moving in three dimensions: in terms of $x_i = (x, y, z)$. Charlie's strategy is then to demand invariance of $L(x_i, \dot{x}_i)$ under three-dimensional rotations and the rescaling symmetry

$$x_i \to \lambda(t) x_i. \tag{2}$$

- C1. Why will Charlie's construction be better than Bob's?
- C2. Use effective theory to argue that the minimal Lagrangian invariant under (2) is

$$L = A \frac{x_i x_i \dot{x}_j \dot{x}_j - (x_i \dot{x}_i)^2}{(x_i x_i)^2}. \tag{3}$$

C3. Explain why the restriction to $z = 1$ can be safely done on (3). Deduce the form of the correct Lagrangian in Bob's coordinates.

The result of Charlie's procedure gives us a sort of "geodesic action" (c.f. Homework 2) for non-relativistic motion on the spatial manifold $\mathbb{RP}^2$. The metric we have found for (almost half of) the two-dimensional sphere (or almost all of $\mathbb{RP}^2$) is called the Fubini-Study metric and is well known to mathematicians (from stereographic coordinates for the sphere).

15 **D:** Dana suggests also starting with Charlie's coordinates $x_i \in \mathbb{R}^3$ but then argues to use a Lagrange multiplier to fix $x_i x_i = 1$, in order to try to restrict to the configuration space $\mathbb{RP}^2$.

D1. Write down Dana's Lagrangian. How many variables/equations will Dana need to solve?

D2. Show how Dana finds the equation of motion[3]

$$\ddot{x}_i = -\dot{x}_j \dot{x}_j x_i. \tag{4}$$

Give a physical interpretation for the right hand side, by thinking about the motion of a particle on a sphere of fixed radius.

Problem 3 (Two-dimensional gravity): A curious feature of two-dimensional gravity (i.e. 1 space, 1 time dimension) is that the only dynamical degree of freedom is the shape of the boundary of the gravitating domain. Effectively, therefore, the theory can be analogous to that of a point particle: the action will depend on the shape of the particle's worldline, or the boundary of the 2d domain.

However this point particle has a very unusual effective action, which we will construct in this problem, for the special case where the 2d gravity theory lives on asymptotically anti-de Sitter (AdS) space. In this limit, the global symmetry of AdS, called $\mathrm{SL}(2,\mathbb{R})$, constrains the effective action for the "particle".

Let $T(\tau)$ denote the worldline of the "particle", with T the coordinate time and τ the proper time. The global symmetry of AdS turns out to lead us to seek an action which is invariant under the following family of nonlinear transformations on T:

$$T(\tau) \to \frac{aT(\tau) + b}{cT(\tau) + d}. \tag{5}$$

where a, b, c, d are real numbers.

We would like is to find a set of variables on which the symmetries of AdS act as a linear transformation – and moreover one where it is easy to write down manifestly invariant terms in an action. It turns out that the best set of variables to work with involve three coordinates u, v, w constrained to obey

$$w^2 - u^2 - v^2 = 1. \tag{6}$$

(We won't explain yet how T is embedded in these three coordinates, but rather will find out how it must be embedded via our construction!). Let us write $X^A = (u, v, w)$. Contract A indices using a "Minkowski" convention, where $X_1^A X_{A2} = u_1 u_2 + v_1 v_2 - w_1 w_2$. The symmetry $\mathrm{SL}(2,\mathbb{R})$ turns out to be the 2+1-dimensional Lorentz group (relativistic symmetries of boosts and rotations) acting in this auxiliary 3-dimensional space. Hence, we can write down invariant objects by simply ensuring that we only contract indices using this relativistic convention.

10 **A:** Using the principles of effective theory, we will write down a Lagrangian for $L(X(\tau))$. Eventually this will tell us the effective action for $T(\tau)$. The first step is to understand how $T(\tau)$ is embedded in these auxiliary coordinates. Since we have 3 coordinates in X^A, but only one variable T, we will need to find 2 constraints.

[3] *Hint:* Take derivatives of $x_i x_i = 1$ to find further constraints on x_i, $\dot{x}_i$, etc.

A1. Argue using effective theory principles that the most natural local constraints are

$$X^A X_A = -1, \tag{7a}$$

$$\dot{X}^A \dot{X}_A = \frac{1}{\epsilon^2}. \tag{7b}$$

We already saw the first one in (6), so you need to justify the second one. Here $\dot{X}^A = \mathrm{d}X^A/\mathrm{d}\tau$. You just need to explain why $\dot{X}^A \dot{X}_A$ is a constant (the form will be useful later).

A2. Show that (6) is satisfied by (here Z is a new coordinate)

$$w = \frac{1 + T^2 + Z^2}{2Z}, \tag{8a}$$

$$v = \frac{1 - T^2 - Z^2}{2Z}, \tag{8b}$$

$$u = \frac{T}{Z}. \tag{8c}$$

A3. Show that (7b) is satisfied by

$$\frac{1}{\epsilon^2} = \frac{\dot{T}^2 + \dot{Z}^2}{Z^2}. \tag{9}$$

Remarkably, this equation reveals (to the initiated eye) how two-dimensional gravity has arisen! The right hand side of this equation is the equation describing the (imaginary time) length (squared) of a worldline in AdS space (c.f. geodesics on Homework 2). We can therefore interpret this constraint as the requirement that *locally* the boundary of the two-dimensional gravity theory maintains a fixed length per unit of proper time.

A4. The theory of interest will look nicest in the limit $\epsilon \to 0$. Explain why in this limit, a consistent solution to (9) is

$$Z = \epsilon \dot{T} + \mathrm{O}\left(\epsilon^3\right). \tag{10}$$

In what follows, neglect all subleading corrections.

10 **B:** Now, we are ready to build the effective action for $T(\tau)$.

B1. The action for two-dimensional gravity will take the form

$$S = \int \mathrm{d}\tau \left[\lambda_1 \left(X^A X_A + 1\right) + \lambda_2 \left(\dot{X}^A \dot{X}_A - \frac{1}{\epsilon^2}\right) + L_{\text{phys}}(X, \dot{X}, \ddot{X}, \ldots)\right], \tag{11}$$

where $\lambda_{1,2}$ are Lagrange multipliers. Use effective theory to find the term(s) in L_{phys} with as few derivatives as possible.

B2. Plug in (8) and (10) into L_{phys}. (You may want to use `Mathematica` for the symbolic manipulations here.) Show that for some constant A,

$$S[T(\tau)] = -A \int \mathrm{d}u \left[\frac{\dddot{T}}{\dot{T}} - \frac{3}{2}\frac{\ddot{T}^2}{\dot{T}^2}\right]. \tag{12}$$

(12) is called the **Schwarzian action**. You could check (but it is not required) that this action is indeed invariant under (5). The point of this problem is that effective theory guarantees for us that this action is invariant under the highly non-trivial symmetries we needed. The quantization of this theory was a very active problem in string theory, about 5 years ago.