Homework 4

Due: September 19 at 11:59 PM. Submit on Canvas.

Problem 1 (Nematic liquid crystal): On Homework 3 we discussed the nematic liquid crystal as a system with configuration space $\mathbb{R}\mathrm{P}^2$. We might also think of such a rodlike molecule as a rotating rigid body where all the mass is concentrated along a single line. Suppose that (in a particular coordinate system) the mass density of the nematic liquid crystal is (here δ denotes the Dirac delta)

$$\rho(z) = \alpha\delta(x)\delta(y)\cdot\begin{cases} 1 & |z| \le L/2 \\ 0 & z > L/2 \end{cases}. \tag{1}$$

20 **A:** We begin by characterizing the inertia tensor(s) of this object.

- A1. Evaluate the tensor/matrix K_{ij} that shows up in the Lagrangian.
- A2. Evaluate the moment of inertia tensor I_{ij}. What are its eigenvalues? You should find two of the eigenvalues of I_{ij} are equal – set these to be $I_1 = I_2$ (in the notation of Lecture 9).

15 **B:** In the absence of external torques, find the most general solution to Euler's equations for a rod-like molecule, where $I_1 = I_2 \neq I_3$.

15 **C:** Now let us analyze this system using Lagrangian mechanics for rigid body motion. Begin with $L = \frac{1}{2}\dot{R}_{ij}\dot{R}_{ik}K_{jk}$ (plus Lagrange multiplier term), as discussed in Lecture 9. What I hope feels surprising is that the configuration space seems to be SO(3), whereas we said before it was $\mathbb{R}\mathrm{P}^2$. If our previous answer was correct, then it must be that secretly the configuration space is *lower dimensional*.

- C1. Consider the 3×3 matrix with components

$$\begin{aligned} &U_{xx}(t) = U_{yy}(t) = \cos\alpha(t), \quad U_{xy}(t) = -U_{yx}(t) = \sin\alpha(t), \\ &U_{zx}(t) = U_{zy}(t) = U_{xz}(t) = U_{yz}(t) = 0, \quad U_{zz}(t) = 1. \end{aligned} \tag{2}$$

  Explain why (or show explicitly) that this matrix is an element of SO(3).

- C2. Show that the Lagrangian L – using K of the nematic liquid crystal! – is invariant under the transformation

$$R_{ij}(t) \to R_{ik}(t)U_{kj}(t), \tag{3}$$

  for *arbitrary* $\alpha(t)$. Explain intuitively why this is happening.

- C3. Explain why the configuration space should effectively not be thought of as SO(3), but rather as a two-dimensional subspace of SO(3) where any two R related by a U above are thought of as the same point in configuration space.[1] To mathematicians, the resulting configuration space is denoted as SO(3)/SO(2). It turns out this space is $\mathbb{R}\mathrm{P}^2$.

[1] *Hint:* Think about Charlie's strategy to deriving an action on $\mathbb{R}\mathrm{P}^2$ on Homework 3. He did not actually need to use a Lagrange multiplier to find an action on $\mathbb{R}\mathrm{P}^2$ starting from a 3-dimensional configuration space!

10 **D:** To understand the last claim above, it helps to use Euler angle coordinates.

- D1. Write down the Lagrangian for a system with $I_1 = I_2$ (and I_3 given by your results from **A**) in terms of (θ, ϕ, ψ).
- D2. Compare to Homework 3, and confirm that L can indeed describe motion on $\mathbb{RP}^2$.
- D3. We have seen that for a zero-thickness rod, the configuration space for rigid body rotation is two-dimensional, $\mathbb{RP}^2$. What is the configuration space, if the rod has finite thickness?

15 **E:** Lastly, let us connect the results of parts **B**, **C** and **D**.

- E1. Find the Euler-Lagrange equations for the Lagrangian of D1.
- E2. Describe the most general possible solution. You don't need to write out an explicit formula, but you should make clear that you understand the physics.[2]
- E3. Sketch the motion of the rod-like molecule with time corresponding to this general solution.
- E4. Discuss the criterion $L_i = I_{ij}\omega_j$ in the body frame coordinates of Euler's equation (and part **B**). Deduce the same conditions on the rigid body's rotation as in E2, thus confirming the consistency between our many approaches to understanding this problem.

Problem 2 (Global shape of SO(3)): In this problem, we will describe an alternative to the Euler angles of Lecture 10 that elucidates the global structure of SO(3).[3] Start with the Lagrangian $L = \frac{1}{2}\dot{R}_{ik}\dot{R}_{ij}K_{jk}$, and assume that $K_{jk} = K_0\delta_{jk}$ for simplicity.

20 **A:** Define the basis of antisymmetric 3×3 matrices

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Suppose we choose $R(t) \in \mathrm{SO}(3)$ such that

$$R(t) = \exp\left[2\alpha(t)\left(\cos\beta(t)J_z + \sin\beta(t)\left[\cos\gamma(t)J_x + \sin\gamma(t)J_y\right]\right)\right]. \tag{5}$$

- A1. Use `Mathematica` (or similar software for symbolic manipulation) to show that

$$L_0 = 4K_0\left(\dot{\alpha}^2 + \sin^2\alpha\dot{\beta}^2 + \sin^2\alpha\sin^2\beta\dot{\gamma}^2\right). \tag{6}$$

- A2. Generalizing (inverting?) the discussion in Lecture 7, argue that the Lagrangian L_0 describes motion on the 3-dimensional sphere S^3, which is defined as the subspace of the 4-dimensional plane $(x, y, z, w) \in \mathbb{R}^4$ obeying $x^2 + y^2 + z^2 + w^2 = 1$. You can use `Mathematica` for algebraic manipulations, but your answer should clearly communicate the physics/math.

5 **B:** It turns out that the $R(t)$ parameterized above completely capture all possible SO(3) matrices, and thus our coordinates completely cover configuration space. However, there is something a little bit peculiar. Find a trajectory $[\alpha(t), \beta(t), \gamma(t)]$ that begins and ends at the same point in configuration space SO(3) – namely, the same R – yet does *not* begin and end at the same point on S^3. Use this construction to suggest that SO(3) must then be identified as S^3 with opposite points identified.

[2] *Hint:* Use the symmetries of the configuration space to choose initial conditions where $\theta = \frac{\pi}{2}$.

[3] This problem is deeply connected to the *quantum* rotation group SU(2), which you can read about in the book (or learn in quantum mechanics).

Analogous to the nematic liquid crystal of Homework 3, SO(3) can be thought of as the set of all lines passing through the origin in $\mathbb{R}^4$, denoted as $\mathbb{RP}^3$.

15 **Problem 3:** In this problem we will discuss the symmetries of "free" rigid body rotation, where the object is fixed to rotate about some pivot point.

1. Starting with $L = \frac{1}{2}\dot{R}_{ik}\dot{R}_{ij}K_{jk}$ (plus Lagrange multiplier), use Noether's Theorem to explicitly construct as many independent conserved quantities as you can think of, for generic K_{jk}; give a transparent physical interpretation of each one.[4]

2. Explain why – although Euler's equations are *first order* equations for three angular velocities $\omega_{1,2,3}$, the existence of the ≥ 3 conserved quantities you hopefully found above does *not* trivialize the dynamics of arbitrary rigid body rotation: namely, it is possible for $\dot{\omega}_{1,2,3} \neq 0$.

[4] *Hint:* You do not need to worry about the Lagrange multiplier term when evaluating Noether's Theorem, just so long as your symmetry transformations are compatible with the constraint that R_{ij} belongs to the configuration space SO(3).