PHYS 5210: Graduate Classical Mechanics **Fall 2022**

Homework 5

Due: September 26 at 11:59 PM. Submit on Canvas.

Problem 1 (Wobbly Earth): The Earth is not a perfect sphere, nor is its mass fully isotropically distributed. As such, we can approximate Earth to be a rigid body which is symmetric about one axis: $I_1 = I_2$, and with

$$\frac{I_3}{I_1} - 1 \sim 3 \times 10^{-3}. \tag{1}$$

The axis the Earth rotates around most quickly is I_3: $\omega_3 \approx 1/(1\text{ day})$.

15 **A:** Let us begin by discussing Euler's equations for a body of this kind.

- A1. Use (your) solutions to Homework 4 to state (without derivation) the most general kind of motion for a rigid body with $I_1 = I_2$.
- A2. Plug in for the actual numbers associated with Earth, and describe the frequency of Earth's wobbling around the 3-axis in its body frame.

5 **B:** For the rest of this problem, we will consider a more non-trivial source of Earth's wobble, due to its gravitational interactions with the Sun. A microscopic calculation of the effect requires analyzing the gravitational potential energy of a non-spherical body, and one finds

$$L = \frac{I_1}{2}\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right) + \frac{I_3}{2}\left(\dot\psi + \dot\phi\cos\theta\right)^2 + \frac{3}{2}(I_3 - I_1)\omega_0^2\cos^2\theta \tag{2}$$

where $\omega_0 \approx 1/(1\text{ year})$ is the orbital period of the Earth.

While fixing the exact prefactor of the last term above requires a microscopic calculation, explain why the function $\cos^2\theta$ that shows up in L is the minimal one which is consistent with either symmetries or physical principles relevant for the problem.

10 **C:** Suppose that the system is on a physical trajectory such that $\theta = \theta_0$ is independent of time.

- C1. Evaluate the Euler-Lagrange equations for L given in (2), and find an equation that constrains the value of θ.
- C2. Following Lecture 11, use the conserved quantity $p_\psi = I_3\omega_3$ to simplify your result from before to an equation relating θ and $\dot\phi$.
- C3. By using the physical values of ω_0, ω_3 and I_3/I_1, argue that the consistent solution to this equation has $\dot\phi$ very small. Estimate it, and thus the period of Earth's precession due to gravitational interactions with the Sun. Compare with the period from part **A**.

10 **D:** Follow our analysis of the spinning top in Lecture 11, and show that we can analyze the motion of Earth's wobble by mapping on to an auxiliary one-dimensional dynamical system, for a particle constrained to $-1 \le z \le 1$, with zero energy, and potential (per mass)

$$V_{\text{eff}}(z) = -\left(1 - z^2\right)\left(a + bz^2\right) + (c - dz)^2. \tag{3}$$

Give expressions for the constants a, b, c, d in terms of $I_1, I_3, \omega_0, \omega_3$.

15 **E:** Now consider more general a, b, c, d. You should ensure that at least in principle the values are physical (e.g. you do not set a parameter that must be positive to in fact be negative!); you can also assume that $I_3 > I_1$, as it is for Earth.

Qualitatively deduce all possible motions of a wobbly planet, by sketching all possible shapes for $V_{\text{eff}}(z)$ (focusing on the number of zeros and where V_{eff} is positive vs. negative).

Problem 2 (A "complex" solid): Consider this microscopic Lagrangian for a one-dimensional solid:

$$L = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2} m \dot{z}_n^2 - \frac{1}{2} K_n (z_n - z_{n+1})^2 \right], \tag{4}$$

where

$$K_n = \begin{cases} A & n \text{ even} \\ B & n \text{ odd} \end{cases}. \tag{5}$$

25 **A:** Let us begin by exactly solving the problem (finding all of the normal modes).

A1. Look for normal mode solutions of the form

$$\begin{pmatrix} x_{2j} \\ x_{2j+1} \end{pmatrix} = \begin{pmatrix} a \\ b\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \mathrm{e}^{2\mathrm{i}\theta j - \mathrm{i}\omega t}. \tag{6}$$

Show that calculating the allowed values of ω^2 reduces to finding the eigenvalues of a 2×2 matrix that depends on θ (and which you should write explicitly). (In this part, please don't ever convert to X, x or k as we did at the end of Lecture 12.)

A2. Find the eigenvalues of this matrix, and thus deduce $\omega_1(\theta)$ and $\omega_2(\theta)$ correspond to two different solutions of the problem, or two different "branches" of dispersion for the vibrational modes.

A3. What are the independent values of θ, corresponding to distinct normal modes?

A4. Suppose $A > B > 0$, and take $|\theta| \ll 1$. Sketch the motion of the atoms on even/odd sites in each of the two types of vibrational normal modes found above. The lower frequency one is called "acoustic", while the higher frequency one is called "optical". This jargon is entirely historical.

A5. Explain what happens if $A = B$.

10 **B:** For each of the two types of vibrational modes found above, qualitatively give the form of the effective (long wavelength) Lagrangian field theory that describes the displacements $X(x, t)$. You do not need to go through as detailed of a derivation as Lectures 12 or 13 – if you can write down the qualitative form of $\mathcal{L}$ and justify it, that is enough for full credit.

10 **C:** Now suppose we looked at an extremely generic one-dimensional solid, where the $K_n > 0$ in (4) are arbitrary. This could model e.g. an amorphous solid like glass. I claim that despite being unable to find exact normal modes based on plane waves, we still know that there is an "acoustic" vibrational mode, which one can find with arbitrarily low frequency ω. Why?[1]

[1] *Hint:* First find a mode which always has $\omega = 0$.

Problem 3 (Rotational-vibrational coupling): In d spatial dimensions, suppose that the motion for a weakly deformable "almost rigid body" can be described as follows. The configuration space consists of d-dimensional orthogonal matrices R with $\det(R) = 1$ (we write $R \in \mathrm{SO}(d)$), together with a single degree of freedom χ which corresponds to the "stretching" of the body.

10 **A:** Using the notation of Lecture 9, the Lagrangian for this almost rigid body is given by

$$L = T(\chi, \dot{\chi}, \dot{R}) - V(\chi) + \Lambda_{ij}\left(R_{ki}R_{kj} - \delta_{ij}\right), \tag{7}$$

where T is the kinetic energy given by

$$T = \int \mathrm{d}^d x' \; \rho(x')v^2, \tag{8}$$

where v^2 denotes the velocity squared, as in Lecture 9. Unlike in Lecture 9, we now propose that the relation between "body frame" coordinates x and fixed "space frame" coordinates x' is given by

$$x'_i(x,t) = \chi(t)R_{ij}(t)x_j. \tag{9}$$

$\chi(t)$ corresponds to a uniform compression/expansion of the body.

A1. Follow the derivation in Lecture 9, but accounting for $\chi(t)$ – and, to the extent it is important, the generality of space dimension d – to find an expression for $T(\chi, \dot{\chi}, \dot{R})$.

A2. Conclude that the rotational R and expansion χ motion are *approximately decoupled* for small amplitude dynamics (e.g. $R_{jk} = \delta_{jk} + \Delta R_{jk}$ and $\chi \approx 1 + \Delta\chi$, with Δ-terms treated as small perturbations).

10 **B:** Restrict to $d = 2$ spatial dimensions. The most general orthogonal matrix in SO(2) takes the form

$$R(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{pmatrix}. \tag{10}$$

B1. Deduce that the configuration space for two-dimensional rotation, called SO(2), is equivalent to S^1.

B2. Find an expression for $L(\theta, \dot{\theta}, \chi, \dot{\chi})$. Do *not* yet restrict to the $\chi \approx 1$ limit.

B3. Explain why even when $\dot{\theta} \neq 0$, the dynamics for χ can be reduced to an effectively one-dimensional problem in Lagrangian mechanics.

5 **C:** Continue with $d = 2$. Now suppose that

$$V(\chi) = \frac{U}{2}(\chi - 1)^2 \tag{11}$$

where U is an extremely large energy scale; namely you can treat $1/U$ as a small parameter. This means that the body's vibrations are highly energetic and the body is nearly rigid.

For a generic rotating nearly-rigid body, would we expect the rotational dynamics to increase or decrease the vibrational frequency?

You might expect the answer to this problem to have relevance for molecular spectroscopy. But, the equivalence between energy and frequency of emitted light in quantum mechanics is important – the shift in ground state energy for the vibrational mode ends up effectively "changing the sign" of the effect above.