Homework 6

Due: October 3 at 11:59 PM. Submit on Canvas.

Problem 1 (Magnetism and spontaneous symmetry breaking): In this problem, we will develop an effective field theory for the magnetization transition in statistical physics. Suppose that we have a material in $d = 3$ spatial dimensions (spatial coordinates denoted with x_i, $i = 1, 2, 3$), and at each point in some microscopic lattice we have an 3-component spin s^α ($\alpha = 1, 2, 3$). Indices $\alpha\beta\cdots$ will denote spin/magnetization components, while $ij\cdots$ denote spatial indices. The continuum coarse-grained magnetization density is defined to be a smooth function with the property that

$$\int_{\text{region } R} \mathrm{d}^3x\, M^\alpha(x) \approx \sum_{\text{lattice sites } v \text{ in region } R} s^\alpha(v). \tag{1}$$

Unlike in Lecture 12, we will not explicitly derive the continuum Lagrangian description for $M^\alpha(x)$, but will instead postulate it from physical principles. There are, a priori, two types of rotations in this problem: rotations of the spatial coordinates x_i, and rotations of the magnetization M^α. We demand invariance of $\mathcal{L}$ under orthogonal transformations $M^\alpha \to R^{\alpha\beta}M^\beta$ of the spins, and under orthogonal transformations $x_i \to Q_{ij}x_j$ of the spatial coordinates. Here R and Q both belong to O(3), but the rotations act on different objects so we will keep them distinct.

20 **A:** The most general Lagrangian density with at most two derivatives is

$$\mathcal{L} = A\left(M^\alpha M^\alpha\right)\partial_t M^\beta \partial_t M^\beta - B\left(M^\alpha M^\alpha\right)\partial_i M^\beta \partial_i M^\beta - C\left(M^\alpha M^\alpha\right), \tag{2}$$

where the (smooth) functions A, B, C are otherwise general.

A1. Explain why the only zero-derivative term that can be written down is C.

A2. Explain why no first derivative terms, either with ∂_t or ∂_i, should be written down.

A3. For the rest of the problem, we will assume that A and B are *constants* (not functions), while

$$C\left(M^2\right) = fM^2 + g\left(M^2\right)^2, \tag{3}$$

with f, g constant as well. Argue that this Lagrangian will (after appropriately rescaling coordinates) have *relativistic invariance* under space/time Lorentz transformations. Henceforth combine $(\partial_t, \partial_i) = \partial_\mu$.

Emergent relativistic invariance is common in field theory, for the symmetry reasons you have found above. For the remainder of the problem, work with

$$\mathcal{L} = -A\partial_\mu M^\alpha \partial^\mu M^\alpha - fM^\alpha M^\alpha - g\left(M^\alpha M^\alpha\right)^2. \tag{4}$$

(Note that we are raising/lowering the spacetime $\mu\nu\cdots$ indices carefully, while the $\alpha\beta$ indices denoting magnetization will always be raised.)

15 **B:** Assume that $A, f, g > 0$, until stated otherwise.

- B1. Find the Euler-Lagrange equations for M^α.
- B2. Show that $M^\alpha = 0$ (for each component) is a solution to these equations.
- B3. Now treat M^α as perturbatively (infinitesmally) small. Show that each component of M^α (approximately) obeys an independent Klein-Gordon-like (massive, relativistic) equation.
- B4. In the limit where M^α is small, what is the approximate form of $\mathcal{L}$? Is your answer consistent with B3?

15 **C:** (4) has a number of continuous symmetries. Let us start with the SO(3) invariance under rotation of the magnetization: $M^\alpha \to R^{\alpha\beta} M^\beta$.

- C1. What are the infinitesimal transformations that generate this symmetry?[1]
- C2. Use Noether's Theorem to deduce conserved currents associated with this symmetry.
- C3. Check that these currents are conserved ("$\partial_\mu J^\mu = 0$") on solutions to the equations of motion.

10 **D:** (4) is also invariant under relativistic translations: $x^\mu \to x^\mu + \epsilon^\mu$. Find the resulting stress-energy tensor $T^{\mu\nu}$ for this theory.

15 **E:** (4) is also invariant under relativistic Lorentz transformations: $x^\mu \to \Lambda^\mu_\nu x^\nu$.

- E1. What is the infinitesimal form of the global transformation above?
- E2. Find the resulting conserved current(s) due to Noether's Theorem.
- E3. Argue that the conservation of these currents imposes that (up to derivative corrections) $T^{\mu\nu} = T^{\nu\mu}$ – namely, the stress-energy tensor should be symmetric. Confirm this is the case.

15 **F:** Now, let us consider $f < 0$.

- F1. Show that the static and *stable* solutions to the Euler-Lagrange equations obey

$$M^\alpha = n^\alpha \sqrt{\frac{|f|}{g}} \tag{5}$$

  where n^α is a t, x-independent unit vector ($n^\alpha n^\alpha = 1$).
- F2. Consider fluctuations around this value of M^α, choosing $n^\alpha = (0, 0, 1)$:

$$M^\alpha = \left(\zeta^1(t,x), \zeta^2(t,x), \sqrt{\frac{|f|}{g}} + \zeta^3(t,x)\right). \tag{6}$$

  Plug in this ansatz for M^α into $\mathcal{L}$, and keep only terms that are at most quadratic order in the fluctuations ζ^α.
- F3. You should have found that two of the ζ^αs obey a *massless* Klein-Gordon equation at quadratic order. Can you think of a reason why?

When $f < 0$, we say that this theory has **spontaneous symmetry breaking** – although $\mathcal{L}$ is invariant under SO(3), the equilibrium state (5) is only invariant under rotations along the n^α-axis (called SO(2)). The existence of massless modes when a continuous symmetry is spontaneously broken is called Goldstone's Theorem. In magnetic systems, when $f > 0$ we have a **paramagnetic** (unordered) phase; when $f < 0$, we have a **ferromagnetic** (ordered) phase, as arises in iron, where the material can spontaneously maintain a magnetic moment.

[1] *Hint:* If stuck, look again at Lectures 5, 8, and/or (solutions to) Problem 3 on Homework 4.

15 **G:** Lastly, let us try to come up with a fully nonlinear effective field theory for the spontaneous symmetry broken phase. The insight is that in (5), it would be more natural to write down an action involving $n^\alpha(x^\mu)$, rather than M^α. As n^α is a unit vector in three-dimensional space, we can think of it as stating that the configuration space for our effective field theory is S^2, the sphere. Previously in class and on homework, we have described how to write down effective theories for dynamics on S^2.

- G1. Predict using effective field theory the simplest possible, lowest derivative, non-trivial $\mathcal{L}$ describing the dynamics of the field $n^\alpha(x^\mu)$. There are multiple plausible ways you could proceed, based on what we have done before.
- G2. Plug in the ansatz (5), but with $n^\alpha(x^\mu)$ a spacetime-dependent object, into (4). Do you find agreement with your answer to G1?

15 **Problem 2 (String theory):** In this problem we will discuss the classical Polyakov action which describes the motion of relativistic strings in string theory. The action describes how a string, parameterized by worldsheet coordinates $(\sigma, \tau) = \sigma^a$, moves through the physical spacetime coordinates X^μ. $\mu\nu\cdots$ indices are raised and lowered with $\eta_{\mu\nu}$. So the fields describing the motion of a string are $X^\mu(\sigma^a)$. At the classical level, the non-trivial action one can write down is

$$S = A \int \mathrm{d}\tau \mathrm{d}\sigma \, \sqrt{-\det(\gamma)}\gamma^{ab}\partial_a X_\mu \partial_b X^\mu \tag{7}$$

Here γ_{ab} is a 2×2 matrix field with no derivatives in the action, while γ^{ab} denotes the inverse of matrix γ_{ab} – we will see that it behaves a bit like a Lagrange multiplier, albeit in a less-than-obvious way. $\det(\gamma)$ denotes the determinant of the 2×2 matrix γ. A is a proportionality constant.

In this problem, we will study the motion of a closed string, where σ is a "periodic" coordinate in which $\sigma = 0$ is identified with $\sigma = L$; hence L is the (coordinate) length of the string.

To actually understand the equations of motion for the relativistic string, it will help to "gauge fix" the action, much like setting $\theta = t$ in our covariant relativistic action for a particle (c.f. Lecture 3). Of course, for the string, this process is rather more involved.

1. Show that S is invariant under the re-scaling $\gamma_{ab} \to \gamma_{ab} C(\sigma, \tau)$ for any C.
2. Show that S is invariant under arbitrary (invertible) reparameterizations: $\tau \to f(\tau, \sigma)$, $\sigma \to g(\tau, \sigma)$. In order to make this work, you will also need to transform γ_{ab} under reparameterization.
3. One can show (but you don't need to) that these symmetries can be used to fix

$$\frac{T + X^1}{\sqrt{2}} = \tau, \tag{8a}$$
$$\partial_\sigma \gamma_{\sigma\sigma} = 0, \tag{8b}$$
$$\det(\gamma) = -1. \tag{8c}$$

where X^1 denotes the first spatial coordinate, while T denotes the time coordinate (sometimes denoted X^0). Conclude that we can write the action in terms of $\gamma_{\sigma\sigma}$ and $\gamma_{\sigma\tau}$ as

$$S = A \int \mathrm{d}\tau \mathrm{d}\sigma \left[\gamma_{\sigma\sigma}(\tau) \left(2\partial_\tau X^- - \partial_\tau X_i \partial_\tau X^i \right) - 2\gamma_{\sigma\tau} \left(\partial_\sigma X^- - \partial_\tau X_i \partial_\sigma X^i \right) + \frac{\left(1 - \gamma_{\sigma\tau}^2\right) \partial_\sigma X_i \partial_\sigma X^i}{\gamma_{\sigma\sigma}} \right] \tag{9}$$

where X_i denotes the transverse spacetime coordinates except for $T$and X^1, and

$$X^- = \frac{T - X^1}{\sqrt{2}}. \tag{10}$$

4. Use an equation of motion, along with any other useful properties stated above, to deduce that γ must be a constant matrix. You can then further impose $\gamma_{\sigma\tau} = 0$.

5. At last, deduce the most general possible solution to the equations of motion of the action. Explain physically the allowed motions of the relativistic string.