PHYS 5210: Graduate Classical Mechanics **Fall 2022**

Homework 7

Due: October 10 at 11:59 PM. Submit on Canvas.

Problem 1 (Lagrangian of charged quantum particle): Consider the Lagrangian (density)

$$\mathcal{L}_0 = \frac{\mathrm{i}\hbar}{2}\left(\bar{\psi}\partial_t\psi - \psi\partial_t\bar{\psi}\right) - \frac{\hbar^2}{2m}\partial_i\psi\partial_i\bar{\psi} \tag{1}$$

Here ψ and $\bar{\psi}$ are to be treated as *independent fields*. $ij\cdots$ indices run over spatial coordinates only.

20 **A:** Show that the Euler-Lagrange equations lead to the free particle Schrödinger equation.

15 **B:** This Lagrangian is invariant under a U(1) symmetry, like the complex Klein-Gordon equation.

- B1. Show that $\mathcal{L}$ is invariant under $\psi \to \psi\mathrm{e}^{\mathrm{i}\lambda q/\hbar}$, $\bar{\psi} \to \bar{\psi}\mathrm{e}^{-\mathrm{i}\lambda q/\hbar}$, for constant λ and q.
- B2. Deduce a conserved current $J^\mu = (J^t, J^i)$, using Noether's Theorem.

20 **C:** Let us try to gauge this symmetry, following Lecture 17.

- C1. Show that the λ transformation from B1 can be made coordinate-dependent if we incorporate a gauge field $A_\mu \to A_\mu + \partial_\mu\lambda$, and replace ∂_μ with a covariant derivative D_μ. Explain how you should choose D_μ to act on both ψ and $\bar{\psi}$.
- C2. Now consider

$$\mathcal{L} = \frac{\mathrm{i}\hbar}{2}\left(\bar{\psi}\mathrm{D}_t\psi - \psi\mathrm{D}_t\bar{\psi}\right) - \frac{\hbar^2}{2m}\mathrm{D}_i\psi\mathrm{D}_i\bar{\psi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2}$$

  Find the Euler-Lagrange equations from varying with respect to $\bar{\psi}$ and A_μ, and comment on their form.

15 **Problem 2:** Take the Lagrangian density $\mathcal{L}$ for relativistic electromagnetism from Lecture 16, and apply Noether's Theorem to gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda(x)$ which leaves $\mathcal{L}$ invariant. What is the resulting conserved current? Briefly comment on what you find.

15 **Problem 3:** In Lecture 16, we developed an effective field theory for relativistic electromagnetism. Explain what would happen if we relaxed the condition that the theory had to be invariant under Lorentz boosts. (Continue to demand invariance under translation and rotation.) Your answer should make sense as the effective field theory for electromagnetism in some ambient medium (like air or water), whose rest frame explicitly breaks the boost invariance of relativity.

Problem 4 (Constraints on particle physics from unitarity): In relativistic quantum field theory, a constraint from unitarity requires that if our Lagrangian density $\mathcal{L}$ is *exact*, then we cannot have any term with more than two derivatives (unless we have infinitely many terms in $\mathcal{L}$). For example, $F_{\mu\nu}F^{\mu\nu}$ is allowed in a theory of electromagnetism, while $(F_{\mu\nu}F^{\mu\nu})^2$ would be forbidden.

This constraint can be used to find strong constraints on Lagrangians in particle physics.

15 **A:** Let us begin by studying theories for spin-1 particles, which are described by fields A_μ. We will introduce an auxiliary field π, defined so that the theory has gauge invariance under

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{3a}$$

$$\pi \to \pi + \lambda. \tag{3b}$$

A1. Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu - \partial_\mu \pi$ are invariant building blocks.

A2. Show that the most general Lagrangian that contains at most two derivatives can be taken to be

$$\mathcal{L} = -\frac{m^2}{2}\left(A_\mu - \partial_\mu \pi\right)\left(A^\mu - \partial^\mu \pi\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4}$$

Here we have taken the liberty of fixing the constants in front of each term (you do not need to justify why this can be done with no loss of generality).

A3. Explain why we can use the "gauge invariance" above to set $\pi = 0$. Thus we find the most general Lagrangian for a massive spin-1 particle!

15 **B:** Now let us determine the quadratic Lagrangian that can describe a massive spin-2 particle, described by a symmetric tensor $h_{\mu\nu} = h_{\nu\mu}$. Now we introduce *two* auxiliary fields: ζ_μ and π, and demand gauge invariance under arbitrary transformations ξ_μ and λ, defined by

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{5a}$$

$$\zeta_\mu \to \zeta_\mu + \xi_\mu + \partial_\mu \lambda, \tag{5b}$$

$$\pi \to \pi + \lambda. \tag{5c}$$

Let us look for $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_2$, where $\mathcal{L}_0$ involves zero derivative terms acting on $h_{\mu\nu}$, and $\mathcal{L}_2$ involves second derivative terms.

B1. Show that the only invariant building block that can enter $\mathcal{L}_0$ is $H_{\mu\nu} = h_{\mu\nu} - \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu + 2\partial_\mu\partial_\nu \pi$.

B2. To have quadratic terms in this building block, we need to find a way to cancel out the fourth-derivative terms in π! Consider

$$\mathcal{L}_0 = \frac{m^2}{2}\left(H_{\mu\nu}H^{\mu\nu} + \theta H_{\mu\nu}H_{\alpha\beta}\eta^{\alpha\beta}\eta^{\mu\nu}\right). \tag{6}$$

Here θ is a constant. Show that we must take $\theta = -1$.

B3. Write down the most general two-derivative Lagrangian density of $h_{\mu\nu}$ (you'll need to think about how to contract the indices!). Show that (up to an overall constant prefactor) there is a *unique* $\mathcal{L}_2$ which is invariant under the gauge transformations above:

$$\mathcal{L}_2 = -\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{2}\partial_\rho h^\mu_\mu \partial^\nu h^\rho_\nu + \frac{1}{2}\partial_\mu h^\mu_\rho \partial_\nu h^{\nu\rho} + \frac{1}{4}\partial_\rho h^\mu_\mu \partial^\rho h^\nu_\nu. \tag{7}$$

It is not enough to simply check this is invariant – you must show there is no other possible functional form.

Combining $\mathcal{L}_0 + \mathcal{L}_2$, we find the most general Lagrangian for a massive spin-2 particle. It is called the Fierz-Pauli Lagrangian.