Homework 8

Due: October 17 at 11:59 PM. Submit on Canvas.

Problem 1 (Elastic solid with cubic symmetry): In this problem, we will derive from first principles the effective theory of a solid whose crystal has *cubic symmetry*. With cubic symmetry, we cannot rotate the σ^I fields arbitrarily: we should pin the σ^I labels on solid cells such that the I unit vectors point along the crystalline axes of the cubic lattice. Thus we will need to re-derive the theory of elasticity, taking into account the reduced symmetries of the system.

10 **A:** Argue that the continuous symmetries of $\mathcal{L}$ will be (here each ϵ denotes an infinitesimally small transformation):

$$t \to t + \epsilon_t, \tag{1a}$$

$$\sigma^I \to \sigma^I + \epsilon^I, \tag{1b}$$

$$X_i \to X_i + \epsilon_i, \tag{1c}$$

$$X_i \to X_i + \epsilon_{ij} X_j, \quad (\epsilon_{ij} = -\epsilon_{ji}). \tag{1d}$$

10 **B:** Which of these continuous symmetries will be spontaneously broken when we look at one equilibrium configuration of the solid: $\sigma^I = X_i \delta_i^I$? Why?

15 **C:** Now, let us deduce the most general possible Lagrangian density $\mathcal{L}$ for a solid with cubic symmetry. The key observation (which you don't need to prove) is that with cubic symmetry, the invariant tensors we can contract IJ indices with are: δ^{IJ}, and the tensor f^{IJKL} defined as

$$f^{IJKL} = \begin{cases} 1 & I = J = K = L \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Follow Lecture 18, and explain why the most general Lagrangian with no more than 4 i-derivatives (or at most 2 derivatives, if t-derivatives) is (here A, B, C, D, E are undetermined constants):

$$\mathcal{L} = A\partial_t\sigma^I\partial_t\sigma^I - B\partial_i\sigma^I\partial_i\sigma^I - C^{IJKL}\partial_i\sigma^I\partial_i\sigma^J\partial_j\sigma^K\partial_j\sigma^L \tag{3}$$

where

$$C^{IJKL} = C\delta^{IJ}\delta^{KL} + \frac{D}{2}\left[\delta^{IK}\delta^{JL} + \delta^{IL}\delta^{JK}\right] + Ef^{IJKL}. \tag{4}$$

15 **D:** Now let us follow Lecture 19 and compute the elastic stress tensor T_{ji}. For simplicity, you can set $B = 0$ for the remainder of this problem.

D1. Use Noether's Theorem to find T_{ij} as a nonlinear function of derivatives of σ.

D2. What physical principle mandates $T_{ij} = T_{ji}$? Check that this condition is obeyed.

20 **E:** Plug in the ansatz $\sigma^I = X_i \delta^I_i + \phi^I$.

E1. Keeping terms to linear order in ϕ^I, show that

$$T_{ij} = (12C + 8D + 4E)\left(\partial_i \phi_j + \partial_j \phi_i\right) - 4(C + D + E)\delta_{ij}\partial_k \phi_k + 8E f_{ijkl}\partial_k \phi_l. \tag{5}$$

E2. Convert from ϕ^I into the more conventional strain tensor u_{ij}, keeping only linear terms in u_{ij}, as in Lecture 19. Conclude that

$$T_{ij} = -\lambda_{ijkl} u_{kl}. \tag{6}$$

You can assume (or justify) $\lambda_{ijkl} = \lambda_{jikl} = \lambda_{ijlk}$. Show that there are three distinct coefficients λ_{xxxx}, λ_{xxyy} and λ_{xyxy} – show how all other non-zero entries of λ_{ijkl} are related to these three by symmetry. Find expressions for these 3 coefficients in terms of C, D, E.

10 **F:** In the previous parts, as in Lecture 19, we derived the stress tensor directly for small displacements ϕ^I. Now, let us find a Lagrangian directly for ϕ^I. Start with (3) (with $B = 0$) and keep only terms quadratic in $\phi_i = \phi^I \delta^I_i$. Show that you find (up to total derivative terms)

$$\mathcal{L} = A \partial_t \phi_i \partial_t \phi_i - F_{ijkl} \partial_i \phi_j \partial_k \phi_l, \tag{7}$$

where the fourth-rank constant tensor $F_{ijkl} = F_{jikl} = F_{ijlk} = F_{klij}$. Show that $F_{ijkl} = \frac{1}{2}\lambda_{ijkl}$.

20 **G:** Lastly, determine how sound waves propagate in an elastic solid with cubic symmetry.

G1. Find the equations of motion for ϕ_i. (There are multiple ways you could do this.)

G2. Plug in the plane wave ansatz $\phi_i = a_i \mathrm{e}^{\mathrm{i}k_j X_j - \mathrm{i}\omega t}$. Describe how to find the most general sound mode by solving some linear algebra problem (which you do not need to do explicitly!).

G3. For simplicity, take $k_Z = 0$. Now, find the dispersion relations $\omega(k_X, k_Y)$ for the three possible sound modes in a cubic solid.

20 **Problem 2 (Thin elastic plates):** A common application of the theory of elasticity is to the dynamics of thin plates and rods, where one dimension of an object is much smaller than the others. In this problem we will focus on the theory of thin plates. Let us assume that in equilibrium, our plate is oriented so that it extends, in equilibrium, from $\sigma_z = -w/2$ to $\sigma_z = w/2$, where w is the thickness of the plate. Assume that the plate has infinite extent in the x, y-directions.

In conventional experiments, the plate is free-standing on its top and bottom ends, so $T_{xz} = T_{yz} = 0$ at the boundaries. We are only interested in dynamics on length scales where $w\partial_x, w\partial_y \ll 1$. Let us therefore approximate that $T_{iz} = 0$ not only at the edges of the plate, but in the interior of the plate as well. We will also write $X \to x$ and $Y \to y$ for convenience.

1. Use that $T_{iz} = 0$ everywhere to show that approximately, if $\phi_z = \phi$ and a, b indices run over x, y only:

$$\phi_a = -z\partial_a \phi + f_a(x, y, t). \tag{8}$$

2. Plug in (8) into $\mathcal{L}$ from (7) (use the λ_{ijkl} for an isotropic solid, for simplicity!). Carefully approximate that $T_{iz} = 0$! Integrate over the z-direction, literally, in the action. Keep only the lowest non-trivial terms in w. Find the effective theory for $\phi(x, y, t)$ and $f_a(x, y, t)$, and interpret physically the result.

3. Find the dispersion relation for ϕ.

4. In an effective field theory of the slowest degrees of freedom, should we keep ϕ or f_a? Based on your answer, discuss the most likely vibrational modes excited if you strike a thin metal plate.