Homework 9

Due: October 24 at 11:59 PM. Submit on Canvas.

Problem 1 (Hamiltonian of a relativistic charged particle): In Lecture 4, we described the Lagrangian for a charged relativistic particle. Focus on the Lagrangian for $x^i(t)$, not the one for $x^\mu(\theta)$!

15 **A:** Carry out the Legendre transform and deduce the Hamiltonian for this system:

$$H = \sqrt{m^2 + (p_i - qA_i)(p_i - qA_i)} + q\Phi \tag{1}$$

where $A_\mu = (\Phi, A_i)$, and we have set $c = 1$ for convenience.

15 **B:** Show that Hamilton's equations of motion reduce to the same equations of motion as in Lecture 4.

Problem 2 (Non-commutative geometry and the Hall effect): In quantum mechanics, it is sometimes useful to think of the spatial coordinates as non-commuting operators. While this might seem quite strange, it actually arises quite naturally in the context of the (quantum) Hall effect. Consider the classical action for a particle of mass m and charge q in a uniform magnetic field of strength B:

$$S = \int \mathrm{d}t \left[\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + qB\dot{y}x - q\Phi(x, y)\right]. \tag{2}$$

10 **A:** Argue that there is a time scale τ (which you should find an expression for, and interpret) such that when the dynamics takes place on time scales $t \gg \tau$, we can actually *ignore the mass term* above – i.e. set $m \approx 0$.[1]

10 **B:** With $m = 0$, S takes the form of the "Hamiltonian action" described in Lecture 23. What is the Hamiltonian? Find a non-trivial Poisson bracket between *spatial coordinates*:

$$[x, y] = -\frac{1}{qB}. \tag{3}$$

This is called **non-commutative geometry**.

10 **C:** Let us describe qualitatively the resulting Hamiltonian dynamics.

C1. Explain why particles will proceed along trajectories of constant Φ.

C2. In a sentence or two, contrast this behavior with that of a usual massive charged particle, e.g. with $B = 0$ but $m \neq 0$. The qualitative discrepancy you should highlight is important for studying physics in strong magnetic fields.

[1] *Hint:* We discussed similar things in Lecture 2, and on Homework 1.

Problem 3 (SO(4) symmetry of the Kepler problem): The Kepler problem (aka, hydrogen atom), with Hamiltonian

$$H = \frac{p_i p_i}{2m} - \frac{k}{r}, \tag{4}$$

with $r = \sqrt{x_i x_i}$, has a hidden symmetry beyond simple rotational invariance (which would be SO(3)). The symmetry of this system is in fact the group SO(4), which has *six* non-commuting conserved quantities.

15 **A:** We first review the Poisson brackets of angular momentum $L_i = \epsilon_{ijk} x_j p_k.$, as discussed in Lecture 24. In this problem, use only $[x_i, p_j] = \delta_{ij}$ and Poisson bracket (commutator) relations from Lecture 24.

- A1. Show that $[L_i, x_j] = \epsilon_{ijk} x_k$.
- A2. Lastly, show that $[L_i, x_j x_j] = 0$. Conclude that $[L_i, f(r) = 0]$ for any function f.
- A3. Using the results $[L_i, p_j] = \epsilon_{ijk} p_k$ and $[L_i, p_j p_j] = 0$ from Lecture 24, show that $[L_i, L_j] = \epsilon_{ijk} L_k$. The identity $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ may help. Alternatively, you can show this component-by-component if you don't want to use the index notation.

10 **B:** Next, we'll need a few more tricks for Poisson brackets involving functions. Show that $[x_i, f(r^2)] = 0$, and

$$\left[p_i, f\left(r^2\right)\right] = -\frac{\mathrm{d}f}{\mathrm{d}(r^2)} 2x_i. \tag{5}$$

In what follows, you can also assume that $[x_i, f(p^2)] = (\mathrm{d}f/\mathrm{d}p^2) 2p_i$.

10 **C:** Next, define the Laplace-Runge-Lenz vector

$$A_i = \epsilon_{ijk} p_j L_k - \frac{mkx_i}{r}. \tag{6}$$

Use the tricks from above to show that $[A_i, H] = 0$.

5 **D:** Show that

$$[A_i, L_j] = \epsilon_{ijk} A_k. \tag{7}$$

You do *not* need to show the commutator

$$[A_i, A_j] = -2mH\epsilon_{ijk} L_k. \tag{8}$$

Since $[A_i, H] = [L_i, H] = 0$, we can treat H as constant for the purposes of evaluating the commutators (algebra) between L_i and A_j. Hence, we find that these commutators between L and A close on themselves, forming a symmetry of the problem. This symmetry is, interestingly, the "same" as the group of rotations in 4 spatial dimensions, called SO(4). This SO(4) symmetry of the hydrogen atom is responsible, in quantum mechanics, for the extensive degeneracy of the non-relativistic Hamiltonian. Since this is an approximation and is broken by relativistic corrections to H, the relativistic corrections break the SO(4) symmetry of hydrogen; therefore, in quantum mechanics, relativistic corrections (fine structure) lift the degeneracy of the spectrum.

Problem 4 (Smectic-A liquid crystal): The smectic-A phase in a liquid crystal behaves as a solid in one of the three spatial dimensions, and a fluid in the other two. This direction is spontaneously chosen, just like the axes in a cubic crystalline solid from Homework 8.

We can build an effective field theory for such a system by combining the ingredients from Lectures 18 and 21 in an appropriate manner. Let $\sigma^I(X_i, t)$ denote the fields corresponding to the location X_i of "fluid cell" σ^I at time t. We orient the internal IJ coordinate axes so that the $I = 3$ direction is aligned with the "solid" direction. For simplicity, let us then denote $\zeta = \sigma^3$ and $\sigma^A = (\sigma^1, \sigma^2)$ – here $AB\cdots$ indices will only run over $A = 1, 2$ – the two directions in which the smectic behaves as a fluid.

Our first step is to impose symmetries on the Lagrangian. We choose

$$
\begin{aligned}
t &\to t + \epsilon_t, && \text{(9a)}\\
X_i &\to X_i + \epsilon_i, && \text{(9b)}\\
X_i &\to R_{ij} X_j && \text{(9c)}\\
\sigma^A &\to \xi^A\left(\sigma^1, \sigma^2\right), \quad \left(\det\left(\frac{\partial \xi^A}{\partial \sigma^B}\right) = 1\right), && \text{(9d)}\\
\zeta &\to \zeta + f\left(\sigma^1, \sigma^2\right), && \text{(9e)}
\end{aligned}
$$

where $R \in \mathrm{O}(3)$ and ξ^A, f are arbitrary (smooth) functions.

5 **A:** Let us begin by deducing the invariant building blocks that make up $\mathcal{L}$.

A1. Show that the most general transformation of $\sigma^I = (\sigma^A, \zeta)$ described above is a volume-preserving coordinate transformation.

A2. Explain why, therefore, $\mathcal{L}$ must be invariant under both volume-preserving transformations in the 12-plane, and in the full 123-space. Deduce that

$$
\begin{aligned}
\mathcal{L} = \mathcal{L}_{\text{fluid}}\Big(&\det\left(\partial_i \sigma^I \partial_i \sigma^J\right), \det\left(\partial_i \sigma^I \partial_i \sigma^J + u^{-2} \partial_t \sigma^I \partial_t \sigma^J\right),\\
&\det\left(\partial_i \sigma^A \partial_i \sigma^B\right), \det\left(\partial_i \sigma^A \partial_i \sigma^B + u^{-2} \partial_t \sigma^A \partial_t \sigma^B\right)\Big). \qquad (10)
\end{aligned}
$$

You should not do any calculations yet, but rather explain conceptually why the answer must be of the form above. As a reminder, the IJ vs. AB indices in the expression above denote whether or not the matrix whose determinant is taken is 3×3 or 2×2, respectively.

5 **B:** Equipped with a Lagrangian, let us now deduce an action for perturbations around equilibrium $\sigma^I = X_i \delta_i^I + \phi^I$. Write $\phi^I = (\phi^A, \eta)$ (namely $\zeta = Z + \eta$), to explicitly separate out the dynamics along the smectic's crystal direction. By expanding out each of the building blocks in (10) up to quadratic order in ϕ^A and/or η, deduce that the quadratic Lagrangian for ϕ^A and θ is

$$
\mathcal{L} = \frac{\rho_1}{2} \partial_t \phi^A \partial_t \phi^A + \frac{\rho_3}{2} \left(\partial_t \eta\right)^2 - \frac{b_1}{2} \left(\partial_A \phi^A\right)^2 - b_2 \partial_A \phi^A \partial_Z \eta - \frac{b_3}{2} \left(\partial_Z \eta\right)^2, \qquad (11)
$$

where $\rho_1, \rho_3, a, b_1, b_2, b_3$ are phenomenological constants within effective field theory. You should assume, without proof, that the Taylor expansions of Lecture 21 up to quadratic order hold in all spatial dimensions (i.e. for any number of fluid-spacetime indices AB vs. IJ), which shortens the calculation.

10 **C:** Let us determine the sound wave dispersion in a smectic-A phase. For simplicity, take $\rho_1 = \rho_3$.[2]

C1. What are the Euler-Lagrange equations for ϕ^A and η?

C2. Plug in a plane wave ansatz for each field, considering the most generic wave number k^I (although of course you can use symmetries to simplify the answer). Show that for some angles there is one sound mode, while at others there are two sound modes.

C3. Use the stability (reality) of $v_{1,2}$ to deduce that $b_1 b_3 \geq b_2^2$ in (11).

[2]Actually, this is not an "assumption", in the sense that you can achieve this by suitable rescalings of Z and η.