PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 1

Principle of least action

August 22

1 State the principle of least action.

Goal: deduce trajectory $x(t)$ of system

↑ Solns to ODE.

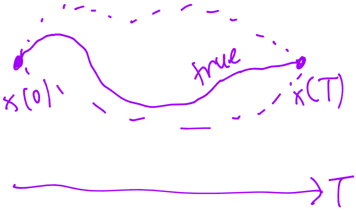
"Old" answer: Solve $F = ma = m\frac{d^2x}{dt^2}$.

"New" answer: P.O.L.A: There exists functional (action) $S[x(t)]$ is extremal (usually minimal) on phys. traj.

functional: $S : \left\{ \text{trajectories } x(t), \text{ fixed BCs } x(0), x(T) \right\} \to \mathbb{R}$ (real #)

Awkward: usually don't know $x(T)$...

In practice: deduce EOM for $x(t)$.



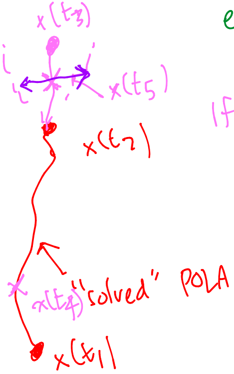
2 Discuss the effective theory approach to physics. What are the consequences of locality in time on the action?

"Effective Theory": semi-rigorous/exhaustive phenom.
what is most generic eqn describing our physical sys...

- ignore microscopic details
- focus: symmetry? slow DOF? principles of Nature

E.g. Locality in time.

e.g. $m\dfrac{d^2x(t)}{dt^2} = F(t)$ [not $m\dfrac{d^2x}{dt^2} = \int_{-\infty}^{t} dt'\, g(t,t')\, x(t')$]



If $S = \dots + x(t_4)\,x(t_5) + \dots$

$$\text{“}\frac{\partial S}{\partial x(t_5)}\text{”} = 0. \qquad = x(t_4) + \dots$$

Lagrangian

Hence: $S = \int dt\, L(x, \dot{x}, \ddot{x}, \dots)$

$\dfrac{dx}{dt} = \dot{x}$

3 Derive the Euler-Lagrange equations. Minimize S?

Warm-up: $S(\underbrace{x(\Delta t)}_{x_1}, \underbrace{x(2\Delta t)}_{x_2}, \dots, \underbrace{x(T-\Delta t)}_{x_N})$.

$$\frac{\partial S}{\partial x_1} = \dots = \frac{\partial S}{\partial x_N} = 0. \quad \text{(algebraic solution)}$$

Idea: functional derivative $\dfrac{\delta S}{\delta x(t)} = 0$. [POLA].

$$S[\bar{x}+\varepsilon g] - S[\bar{x}] = \int_0^T dt \left[L(\bar{x}+\varepsilon g, \dot{\bar{x}}+\varepsilon \dot{g}) - L(\bar{x}, \dot{\bar{x}})\right]$$

$$= \int_0^T dt \left[\left.\frac{\partial L}{\partial x}\right|_{\bar{x}} \varepsilon g + \varepsilon \dot{g} \left.\frac{\partial L}{\partial \dot{x}}\right|_{\bar{x}}\right]$$

$$= \int_0^T dt\, \varepsilon g \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] + \cancel{\varepsilon g(T) \left.\frac{\partial L}{\partial \dot{x}}\right|_T - \varepsilon g(0) \left.\frac{\partial L}{\partial \dot{x}}\right|_0}$$

$g(0) = g(T) = 0$.

Taylor expanding:

$$S[\bar{x}+\varepsilon g] = S[\bar{x}] + \varepsilon \int_0^T dt\, g(t) \frac{\delta S}{\delta x(t)} + O(\varepsilon^2)$$

$$\boxed{\frac{\delta S}{\delta x(t)} = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0}$$

4 Extend to systems with multiple degrees of freedom.

$$S[x_1(t), \dots, x_n(t)] = \int dt\, L(\underbrace{x_1, \dots, x_n}_{x_i}, \underbrace{\dot{x}_1, \dots, \dot{x}_n}_{\dot{x}_i})$$

n particles in 1d or

1 particle in n-d ...

Instead of $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$ ...

$$S[x_i(t)] = \int dt\, L(x_i, \dot{x}_i)$$

$$\frac{\delta S}{\delta x_i(t)} = 0 = \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}$$