PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 10

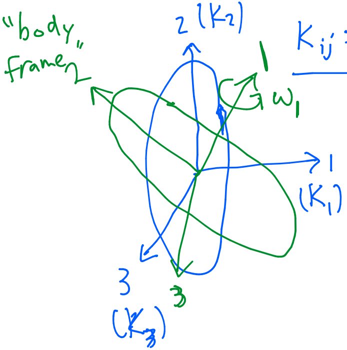
Euler angles

September 14

1 Review Euler's equations.

Euler's equations:

$$\boxed{\begin{aligned} I_1 \dot{\omega}_1 &= (I_2 - I_3)\,\omega_2 \omega_3 \\ I_2 \dot{\omega}_2 &= (I_3 - I_1)\,\omega_3 \omega_1 \\ I_3 \dot{\omega}_3 &= (I_1 - I_2)\,\omega_1 \omega_2 \end{aligned}}$$



$K_{ij} = \int \rho x_i x_j$

$$\boxed{\begin{aligned} I_{ij} &= \delta_{ij} K_{ll} - K_{ij} \\ I &= 1 \cdot \mathrm{tr}(K) - K \end{aligned}}$$

$$K_1 + K_2 = I_3$$
$$K_1 + K_3 = I_2$$
$$K_2 + K_3 = I_1$$

$$\vec{L} = I \vec{\omega}$$

↑ mon. inert.

$\boxed{2}$ Overview the Poinsot construction.

$I_{1,2,3}$ = eig. of moment of in.

(kinetic) energy:

$$E = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

(angular momentum)^2

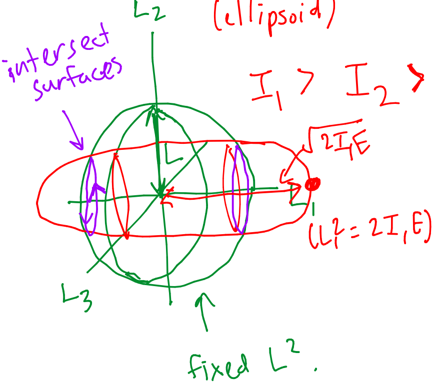
$$L^2 = L_1^2 + L_2^2 + L_3^2$$

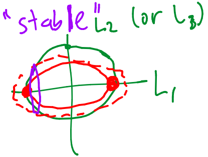
$$= (I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2$$

$$\dot{E} = 0 \quad \hookrightarrow E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}$$

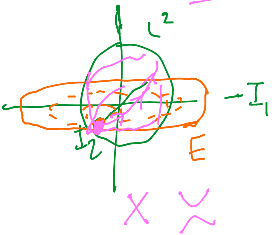
(ellipsoid)

$I_1 > I_2 > I_3$





unstable motion



3 Derive the stability condition from Poinsot directly from Euler's equations.

Suppose
$$\omega_2 = \Omega + \underline{\delta\omega_2(t)}$$
$$\omega_1 = \underline{\delta\omega_1(t)}$$
$$\omega_3 \quad \underline{\delta\omega_3(t)}$$

infinites. perturb.

LINEARIZE Euler's equations:

$$I_1 \delta\dot{\omega}_1 = (I_2 - I_3)(\Omega + \cancel{\delta\omega_2})\delta\omega_3$$
$$I_2 \delta\dot{\omega}_2 = (I_3 - I_1)\cancel{\delta\omega_1 \delta\omega_3} = 0$$
$$I_3 \delta\dot{\omega}_3 = (I_1 - I_2)\delta\omega_1(\Omega + \cancel{\delta\omega_2})$$

$$\delta\ddot{\omega}_1 = C \cdot \delta\omega_1,$$
$$C = \frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_3}$$

$I_2 - I_3 > 0$ and
$I_1 - I_2 > 0$

$$r^2 = C > 0$$

$$\delta\omega_1 = A_1 e^{rt} + A_2 e^{-rt}$$

const. int.

r is real.

4 Define and sketch the Euler angles.

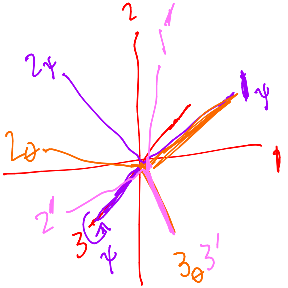
Euler angles = coordinates on $SO(3)$: (θ, ϕ, ψ)

$$x'_i = R_{ij} x_j$$

"space frame" (x'_i)   "body frame" (x_j)

$$R = R_\phi R_\theta R_\psi$$

fix 3
$$R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



fix 1_ψ
$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

fix 3_θ
$$R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5 Write out R in terms of Euler angles.

$$R = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\theta\sin\phi \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

$$\left[\frac{1}{2}\mathrm{tr}\left(\dot{R}K\dot{R}^T\right)\right]$$

Find body frame $\omega_1, \omega_2, \omega_3$:

$$\dot{R} = R\Omega \quad \rightarrow \quad R^{-1}\dot{R} = \Omega$$

$$L = \frac{1}{2}\dot{R}_{ij}\dot{R}_{ik}K_{jk} + \cancel{\cdot}$$

Euler angles restrict to SO(3) no Λ.

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

$$L = \frac{1}{2}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right]$$