PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 11

The spinning top

September 16

$\boxed{1}$ 3 Write down the Lagrangian for the spinning top.

potential energy:

$$V = Mgh$$
$$= Mg\ell\cos\theta$$

kinetic energy:

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

Assume $I_1 = I_2$ and use Euler angles:

$$T = \frac{1}{2}I_1\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + \frac{1}{2}I_3\left(\dot\psi + \dot\phi\cos\theta\right)^2$$

$$L = T - V$$

$$L = \frac{1}{2}I_1\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right) + \frac{1}{2}I_3\left(\dot\psi + \dot\theta\cos\theta\right)^2 - Mg\ell\cos\theta$$

2 What are the conserved quantities?

$$L = \frac{1}{2} I_1 \left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2\right) + \frac{1}{2} I_3 \left(\dot\psi + \dot\phi\cos\theta\right)^2 \overset{(+)}{-} Mgl\cos\theta$$

Begin w/ conserved quantities?

#1: $\psi \to \psi + \varepsilon$ $[g_\psi = 1]$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3$$

#2: $\phi \to \phi + \varepsilon$ $[g_\phi = 1]$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\, \dot\phi + \cos\theta \cdot p_\psi$$

#3: $t \to t + \varepsilon$ $[T = 1]$

$$E = T + V$$

Write E in terms of p_ψ, p_ϕ:

$$E = Mgl\cos\theta + \frac{1}{2} I_1 \dot\theta^2 + \frac{p_\psi^2}{2I_3} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta}$$

Write $z = \cos\theta$

$\dot z = -\sin\theta\, \dot\theta$

$$E = Mglz + \frac{p_\psi^2}{2I_3} + \frac{(I_1\dot z)^2 + (p_\phi - p_\psi z)^2}{2I_1(1-z^2)}$$

3 Reduce the problem to an effective 1d Newtonian mechanics problem.

$$E = Mglz + \frac{p_\psi^2}{2I_3} + \frac{(I_1\dot z)^2 + (p_\phi - p_\psi z)^2}{2I_1(1-z^2)}$$

$$0 = \dot z^2 + (b - az)^2 - (1-z^2)(\alpha - \beta z)$$

$$\alpha = \frac{1}{I_1}\left(2E - \frac{p_\psi^2}{2I_3}\right)$$

$$\beta = \frac{2Mgl}{I_1}$$

$$a = \frac{p_\psi}{I_1}$$

$$b = \frac{p_\phi}{I_1}$$

[optional: $\tau = at$:

$$0 = \left(\frac{dz}{d\tau}\right)^2 + \left(\frac{b}{a} - z\right)^2 - (1-z^2)\left(\frac{\alpha}{a^2} - \frac{\beta z}{a^2}\right)$$]

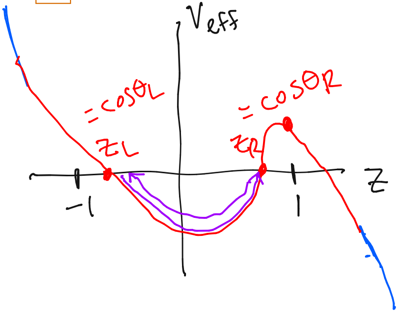
"$E' = T_z + V_z$"

"Particle of mass $m=2$, $E'=0$":

$$E' = 0 = \dot z^2 + V_{eff}(z)$$

$$V_{eff}(z) = (b - az)^2 - (1-z^2)(\alpha - \beta z)$$

4 Discuss qualitatively the possible solutions to this problem.

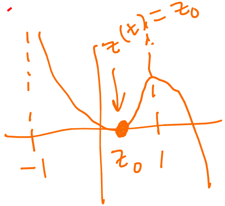


$$V_{eff}(z) = \underbrace{(b-az)^2}_{>0} - \underbrace{(1-z^2)}_{=0 \text{ at } z=1}(\alpha-\beta z)$$

$$V_{eff}(z\to\infty): -\beta z^3$$

Not all α, β, a, b correspond to physically allowed.

Possible:



5 Discuss the limit of a rapidly spinning top.

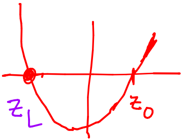
Initial conditions: $z = z_0$ $\quad \boxed{\dot{\phi} = 0}$ at $t = 0$.

$$p_\phi = p_\psi z_0 \quad \text{or} \quad b = a z_0.$$

$$E = \frac{1}{2} I_3 \omega_3^2 + Mgl z_0, \quad \text{or} \quad \alpha = \beta z_0.$$

$$V_{eff}(z) = (b - az)^2 - (1 - z^2)(\alpha - \beta z)$$
$$= (z - z_0)\left(a^2(z - z_0) + \beta \underbrace{(1 - z^2)}_{>0 \ \approx \text{const.}}\right)$$

$V_{eff} \approx$ harmonic



Top spinning fast: $\quad a^2 \gg \beta$

$(p_\psi^2) \quad (Mgl)$

$$0 = a^2(z - z_0) + \beta(1 - z_0^2)$$

$$z_R = z_0 \qquad z_L = z_0 - \frac{\beta}{a^2}(1 - z_0^2)$$

$$\cos\theta_R - \cos\theta_L \approx \sin\theta_R \Delta\theta$$
$$= \frac{2MglI_1}{(I_3\omega_3)^2}\sin^2\theta_0$$

θ oscillate frequency:

$$V_{eff} \sim a^2 (z - z_{min})^2$$

$$\omega_{osc} = \sqrt{\frac{2 \cdot a^2}{2}} = a = \frac{I_3 \omega_3}{I_1}$$