PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 12

Vibrations of solids

September 19

1 Write down the Lagrangian, and equations of motion, for a one-dimensional solid made of masses and springs.

$$\cdots + (z_n - z_{n+1})^2 + (z_{n-1} - z_n)^2$$

z_n = disp. of block n rel. to equilib.

$$L = \sum_{j=-\infty}^{\infty} \frac{1}{2} m \dot{z}_j^2 - \sum_{j=-\infty}^{\infty} \frac{1}{2} m\omega_0^2 (z_j - z_{j+1})^2$$

[Position of block n:

$$X_n = \underbrace{z_n}_{\text{disp.}} + \underbrace{na}_{\text{equilib. dist.}}$$
]

$$\frac{\partial L}{\partial z_n} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}_n} = 0$$

$$-\cancel{m}\omega_0^2 \left[(z_n - z_{n+1}) + (z_n - z_{n-1}) \right] - \cancel{m}\ddot{z}_n = 0$$

$$\ddot{z}_n = -\omega_0^2 \left[2z_n - z_{n-1} - z_{n+1} \right].$$

2 Sketch how we solve the coupled oscillator problem.

$$L = \frac{1}{2} m \dot{z}_j \dot{z}_j - \frac{1}{2} m C_{jk} \underbrace{z_j z_k}_{\sum_j (z_j - z_{j+1})^2 \ldots}$$

$z_k z_j$: $C_{jk} = C_{kj}$

($\sum_j$: repeated index)

E-L:

$$\ddot{z}_j = -C_{jk} z_k \qquad \left[\ddot{\vec{z}} = -C\vec{z}\right]$$

change of basis to diagonalize C:

e-val/vector of C are labeled by θ.

$$C\vec{u}_{(\theta)} = \omega^2_{(\theta)} \vec{u}_{(\theta)}$$

$\omega^2_{(\theta)} > 0$: stability of $\vec{z} = \vec{0}$.

EOMs: one solution: $\vec{z}(t) = \vec{u}_{(\theta)} \left[A \cos(\omega_{(\theta)} t) + B \sin(\omega_{(\theta)} t)\right]$

General: $\vec{z}(t) = \sum_\theta \vec{u}_{(\theta)} \left[A_{(\theta)} \cos(\omega_{(\theta)} t) + B_{(\theta)} \sin(\omega_{(\theta)} t)\right]$.

3 Write down the normal modes for our solid problem.

In our problem:
$$C_{jk} = \begin{cases} 2\omega_0^2 & j=k \\ -\omega_0^2 & j=k-1 \text{ or } k+1 \\ 0 & \text{otherwise} \end{cases}.$$

"Try" eigenvector: $(\vec{u}_\theta)_j = e^{i\theta j}$ ← discretization of plane wave: e^{ikx}

$$[C\vec{u}_{(\theta)}]_j = -\omega_0^2 \vec{u}_{(\theta),j-1} + 2\omega_0^2 \vec{u}_{(\theta),j} - \omega_0^2 \vec{u}_{(\theta),j+1}$$
$$= e^{i\theta j}\left[-\omega_0^2 e^{-i\theta} + 2\omega_0^2 - \omega_0^2 e^{i\theta}\right]$$
$$= u_{(\theta)j}\left[\underbrace{2\omega_0^2 - 2\omega_0^2\cos\theta}_{\text{eigenvalue}}\right]$$

$$= -\omega_0^2 e^{i\theta(j-1)} + 2\omega_0^2 e^{i\theta j} - \omega_0^2 e^{i\theta(j+1)}$$

$$e^{i\theta(j-1)} = e^{i\theta j} e^{-i\theta}$$

$$\omega_{(\theta)}^2 = 4\omega_0^2 \sin^2\left(\frac{\theta}{2}\right)$$

$$\omega_{(\theta)} = 2\omega_0 \sin\frac{\theta}{2}$$

4 What is the general solution for $z_j(t)$?

$z_j(0) u^*_{(\theta),j} = \tilde{z}_{(\theta)}(0)$ [Initial conditions]

" $\langle u_{(\theta)} | z \rangle$ " [and $\dot{\tilde{z}}_{(\theta)}(0) = \dot{z}_j(0)\, u^*_{(\theta),j}$]

Like Schrodinger equation:

$\tilde{z}_{(\theta)}(t) = A_{(\theta)} \cos(\omega_{(\theta)} t) + B_{(\theta)} \sin(\omega_{(\theta)} t)$

$\tilde{z}_{(\theta)}(0) = A_{(\theta)}$ $\quad \dot{\tilde{z}}_{(\theta)}(0) = \omega_{(\theta)} B_{(\theta)}$

General solution: " $|z\rangle = \sum |u_{(\theta)}\rangle e^{-\cdots}$ "

becomes $z_j(t) = \sum_\theta \tilde{z}_{(\theta)}(t)\, u_{(\theta),j}$

$\vec{z}(t)^* = \vec{z}(t) = \sum_\theta \tilde{z}_{(\theta)}(t)\, \vec{u}_{(\theta)}$ complex

Reality of $z_j(t)$: $\tilde{z}_{(\theta)}(t) = \tilde{z}^*_{(-\theta)}(t)$

5 Discuss the continuum limit.

General solution: $\sum_{\theta} \tilde{z}_{(\theta)}(t)\vec{u}_{(\theta)}$

Why?

$(u_\theta)_j = e^{i\theta j}$

$= e^{i(\theta+2\pi)j}$

$= e^{i\theta j} \cancel{e^{2\pi i j}}\ 1$

b/c j is an integer.

Claim: θ take any value from $-\pi < \theta \leq \pi$.

$\left(-\pi/a < k \leq \pi/a\right)$

$$z_j(t) = \int_{-\pi}^{\pi} d\theta\, \tilde{z}_\theta(t) e^{i\theta j}$$

eq. spacing btwn blocks

Suppose $x = ja$ is used to denote each block.. $z_x \to z(x)$

Actual block position: $X(x) = z(x) + x$

physical locat

displacement.

"wave number":

Replace: $j = x/a$. $\theta = ak$

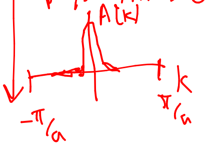
$$X(x,t) = x + \int_{-\pi/a}^{\pi/a} dk \cdot a\, z(k,t) e^{ikx} = \int_{-\pi/a}^{\pi/a} dk \left[A(k) e^{ikx + i\omega(k)t} + B(k) e^{+ikx - i\omega(k)t}\right] + x$$

6 Discuss the Lagrangian in the continuum limit.

$$X(x,t) - x = \int_{\cancel{-\pi/a}\ -\infty}^{\cancel{\pi/a}\ \infty} dk \left[A(k)\, \underbrace{e^{ikx + i\omega(k)t}}_{\text{left-moving dispers. wave:}} + B(k)\, \underbrace{e^{ikx - i\omega(k)t}}_{\text{right-moving wave}} \right]$$

$$v_g = \frac{d\omega}{dk} \qquad \lambda = 2\pi/k$$

continuum limit: phys. init. cond.



Let's look Lagrangian:

$$L = \int_{\cancel{-\pi/a}\ -\infty}^{\cancel{\pi/a}\ \infty} dk \left[\underbrace{\frac{m}{2}\left|\frac{\partial \tilde{z}}{\partial t}\right|^2}_{\frac{m}{2}\sum \dot{z}_j^2} - \underbrace{\frac{m}{2}\, \overset{a^2\omega_0^2k^2}{\cancel{\omega(k)^2}}\, |\tilde{z}(k,t)|^2}_{\text{diagonalized } C_{jk} z_j z_k} \right]$$

$a \to 0$

v_s fixed

is continuum limit

$\overline{v_s}$

$$\omega(k) = 2\omega_0 \sin\frac{ka}{2} \approx 2\omega_0 \cdot \frac{ka}{2} = (a\omega_0)k$$

$$L = \int dx\, \frac{m}{2}\left[\left(\frac{\partial z}{\partial t}\right)^2 - \overbrace{(a\omega_0)^2}^{v_s^2} \left(\frac{\partial z}{\partial x}\right)^2\right]$$