PHYS 5210

Graduate Classical Mechanics

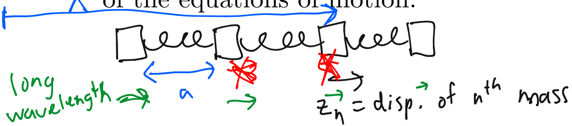
Fall 2022

Lecture 13

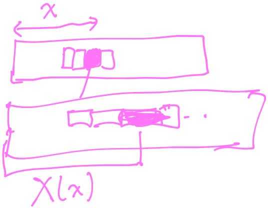
Lagrangian field theory and sound waves

September 21

1 Review the coupled oscillator problem, and discuss the continuum limit of the equations of motion.



$$\ddot{z}_n = -\omega_0^2\left[2z_n - z_{n-1} - z_{n+1}\right]$$



Goal: continuum description:

pos dynamically $\rightarrow$ $X(x,t) = x + z_n(t)$

if $x = na$ — pos at rest

$$\ddot{z}_n = \frac{\partial^2}{\partial t^2}\left[X(x,t) - \cancel{x}\right]$$

$$2z_n - z_{n-1} - z_{n+1}$$
$$= 2\left[X(na) - \cancel{na}\right] - \left[\underbrace{X((n-1)a)} - \cancel{(n-1)a}\right]$$
$$- \left[\underbrace{X((n+1)a)} - \cancel{(n+1)a}\right]$$

$$X(na+a) \approx X(na) + a\frac{\partial X}{\partial x} + \frac{a^2}{2}\frac{\partial^2 X}{\partial x^2}$$

$$= -a^2\frac{\partial^2 X}{\partial x^2}$$

Summary:

$$\frac{\partial^2 X}{\partial t^2} = \underbrace{(a\omega_0)^2}\frac{\partial^2 X}{\partial x^2}$$

Call $a\omega_0 = v_s$ (speed of sound)

Continuum limit: v_s fixed, $a \to 0$.

2 Review the coupled oscillator Lagrangian, and describe its continuum limit.

$$L = \sum_n \left[\frac{1}{2} m \dot{z}_n^2 - \underbrace{\frac{1}{2} m \omega_0^2 (z_n - z_{n+1})^2}_{} \right]$$

Plug in:

$$X(\underbrace{na}_{x}, t) = x + z_n(t)$$

$$\sum_n = \int \frac{dx}{a}$$

$$\frac{1}{2} m \left(\frac{\partial X}{\partial t} \right)^2 \Bigg|_{x=na}$$

$$z_n - z_{n+1} = [X(na) - na] - [X((n+1)a) - (n+1)a]$$

$$= a - a \frac{\partial X}{\partial x} + \cdots$$

$$\frac{\partial}{\partial x}(X - x)$$

$\frac{\text{mass}}{\text{length}}$ $\quad \rho = \frac{m}{a}$

Plug in: $L = \int dx \left[\frac{\rho}{2} \left(\frac{\partial X}{\partial t} \right)^2 - \frac{\rho}{2} v_s^2 \left(1 - \frac{\partial X}{\partial x} \right)^2 \right]$

kinetic $\qquad$ potential

"field theory"

$$S = \int dt\, L = \int dt\, dx \ \mathcal{L}$$

Lagrangian "density"

$$\mathcal{L} = \frac{\rho}{2} \left[\left(\frac{\partial X}{\partial t} \right)^2 - v_s^2 \left(1 - \frac{\partial X}{\partial x} \right)^2 \right]$$

$X(x,t)$ is a "field"

$$\frac{\partial X}{\partial t} = \partial_t X, \quad \frac{\partial X}{\partial x} = \partial_x X$$

3 Describe the formalism of Lagrangian field theory.

EFT = effective field theory

philosophy: - write down anything in $\mathcal{L}$ (S) allowed by symmetry

- ignore terms w/ "too many derivatives"

New: degree of freedom $X(x, t)$

x: space, t: time

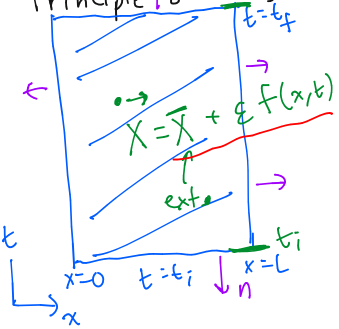
Action S will be a functional $S[X[x,t]]$

Locality in space & time:

$$\boxed{S = \int dt\, dx\, \mathcal{L}(X, \partial_t X, \partial_x X, \partial_t \partial_x X, \ldots)}$$

4 What are the generalization of the Euler-Lagrange equations?

Principle of least action: $\boxed{\dfrac{\delta S}{\delta X(x,t)} = 0}$ (on physical trajectories)



$$0 = \frac{dS}{d\varepsilon}\bigg|_{\varepsilon=0} = \int dx\,dt\, \frac{\delta S}{\delta X(x,t)}\, f(x,t)$$

$$\int dt\,dx \left[\frac{\partial \mathcal{L}}{\partial X} f + \frac{\partial \mathcal{L}}{\partial(\partial_t X)}\partial_t f + \frac{\partial \mathcal{L}}{\partial(\partial_x X)}\partial_x f\right]$$

$$= \int dt\,dx \left[\frac{\partial \mathcal{L}}{\partial X} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t X)} - \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x X)}\right] f$$

$$\cancel{+ \oint f \left[\frac{\partial \mathcal{L}}{\partial(\partial_t X)} \cdot n_t + \frac{\partial \mathcal{L}}{\partial(\partial_x X)} n_x\right]}$$

$f=0$ on bdy

Postulate:

$f = 0$ at boundaries $[t = t_i, t_f$; $x = 0, L]$

Euler-Lagrange: $\dfrac{\delta S}{\delta X} = \dfrac{\partial \mathcal{L}}{\partial X} - \partial_t \dfrac{\partial \mathcal{L}}{\partial(\partial_t X)} - \partial_x \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_x X)}\right) = 0$

5 What are the continuum equations of motion for the coupled oscillator problem?

$$\mathcal{L} = \frac{\rho}{2}\left[\left(\overset{\partial_t X}{\frac{\partial X}{\partial t}}\right)^2 - v_s^2\left(1 - \overset{\partial_x X}{\frac{\partial X}{\partial x}}\right)^2\right]$$

Euler-L: $$0 = -\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t X)} - \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x X)}$$

$$= -\partial_t\left[\rho\, \partial_t X\right] - \partial_x\left[-\rho v_s^2\left(\partial_x X - \cancel{1}\right)\right]$$

$$\hookrightarrow \cancel{\rho}\, \partial_t^2 X = \cancel{\rho}\, v_s^2 \partial_x^2 X$$ wave equation!

If $\mathcal{L} \to \mathcal{L} + \partial_x F^x + \partial_t F^t$, for any F_x & F^t:
E-L eqns don't change.

$$\left.S \to S + \underbrace{\int dx\, dt \left[\partial_t F^t + \partial_x F^x\right]}_{\nabla \cdot F}\right\}$$ is boundary term by div thm

$\boxed{6}$ Discuss how to solve the wave equation in terms of plane waves.

$$\mathcal{L} = \frac{\rho}{2}\left[(\partial_t X)^2 - v_s^2\left((\partial_x X)^2 - 2\cancel{\partial_x X} + \cancel{1}\right)\right]$$

$\rightarrow$ const. doesn't affect EOM.

$\partial_x(-\rho v_s^2 X)$ is a total der.

EOM: $\partial_t^2 X = v_s^2 \partial_x^2 X$ "wave equation"

Look for plane wave sol'ns: $X = e^{ikx - i\omega t}$

$$(-i\omega)^2 \cancel{e^{ikx-i\omega t}} = v_s^2 (ik)^2 \cancel{e^{ikx-i\omega t}}$$

$\omega^2 = v_s^2 k^2$, so $\omega = v_s k$ or $-v_s k$.

General solution:

$$X(x,t) = \int dk \left[A(k) e^{ikx + iv_s kt} + B(k) e^{ikx - iv_s kt}\right]$$

FT F.T.

$$= \tilde{A}(x + v_s t) + \tilde{B}(x - v_s t).$$