PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 14

Noether's Theorem for fields

September 23

1 Extend Lagrangian field theory to higher dimensions.

$$S = \int dt\, dx\, \mathcal{L}(\phi, \partial_t \phi, \partial_x \phi) \qquad \left(\partial_t \phi = \frac{\partial \phi}{\partial t}\right)$$

Principle of least action:

$$0 = \frac{\delta S}{\delta \phi(x,t)} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} - \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)}$$

In higher dimensions: $x^\mu = (t, x, y, z, \ldots)$ $\qquad \underline{\frac{\partial \phi}{\partial x^\mu} = \partial_\mu \phi}$

$$\to \quad S = \int d^D x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a, \ldots)$$

a = different kinds of fields

μ = spacetime index

$$\sum_\mu \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = \frac{\partial}{\partial t} \cdots + \frac{\partial}{\partial x} \cdots + \cdots$$

Euler-L:

$$0 = \frac{\delta S}{\delta \phi_a(x)} = \frac{\partial \mathcal{L}}{\partial \phi_a} - \underbrace{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}}$$

2 State Noether's Theorem for fields.

Suppose S is invariant:

$$\phi_a \to \phi_a + \varepsilon g_a \, ; \quad x^\mu \to x^\mu + \varepsilon Y^\mu \, ; \quad \mathcal{L} \to \mathcal{L} + \varepsilon \overbrace{\partial_\mu K^\mu}^{\text{total divergence}}$$

Noether's Thm: $\partial_\mu J^\mu = 0$ on physical trajectories, where

(state w/o proof)

$$J^\mu = \mathcal{L} Y^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a Y^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} g_a + K^\mu$$

Before: conserved Q $\left(\frac{dQ}{dt} = 0 \text{ on. phys. / ext. traj} \right)$

$$\partial_\mu J^\mu = \underbrace{\frac{\partial J^t}{\partial t} + \frac{\partial J^x}{\partial x} = 0}$$

- Conservation

+

- spacetime locality

Conserved quantity?

$$Q = \int dx \, J^t \, ?$$

$$\frac{dQ}{dt} = \int dx \, \partial_t J^t = -\int dx \, \partial_x J^x$$

$$= \cancel{-J_x(\infty)} + \cancel{J_x(-\infty)} = 0.$$

3 What is the physical origin of $X(x,t) \to X(x,t) + \epsilon$ symmetry in a solid? What is the consequence?

Recall: 1d solid, block started at x moves to $X(x,t)$

$$\mathcal{L} = \frac{\rho}{2}\left[(\partial_t X)^2 - v_s^2 (\partial_x X)^2\right]$$

Notice: $\mathcal{L}$ invariant if $X \to X + \varepsilon$ $[g=1]$

Why? global displacement cost no energy (universe agnostic to $X=0$).

Noether's Thm: $J^\mu = (J^t, J^x)$

$$J^t = \frac{\partial \mathcal{L}}{\partial(\partial_t X)} = \underbrace{\rho \partial_t X}_{\text{momentum density (?)}}$$

$$J^x = \frac{\partial \mathcal{L}}{\partial(\partial_x X)} = -\rho v_s^2 \partial_x X$$

E-L eqn.

$$\partial_\mu J^\mu = 0 = \partial_t J^t + \partial_x J^x = \rho\left[\partial_t^2 X - v_s^2 \partial_x X\right]$$

4 What is the consequence of time-translation symmetry?

$\left[\rho = \frac{m_{micro}}{a_{micro}}\right]$

Symmetry: $t \to t + \varepsilon$ $\quad [Y^t = 1,\ Y^x = 0]$

$$J^t_{(TT)} = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_t X)}\partial_t X = -\frac{\rho v_s^2}{2}(\partial_x X)^2 - \frac{\rho}{2}(\partial_t X)^2$$

$= -$ [energy dens.] $= -$pot. $-$ kin.

$$J^x_{(TT)} = -\frac{\partial \mathcal{L}}{\partial(\partial_x X)}\partial_t X = \rho v_s^2 \partial_x X \partial_t X$$

↳ t, x components of energy (t) component of stress-energy (energy-mom.) tensor.

energy

t-comp

$-J^t_{(TT)} \to T^{tt}$

$-J^x_{(TT)} \to T^{xt}$

$$\partial_t T^{tt} + \partial_x T^{xt} = 0\ ? = \rho\, \partial_t X \big[\underbrace{\partial_t^2 X - v_s^2 \partial_x^2 X}_{=0\ !}\big]$$

"on shell"

$$\partial_t T^{tt} = \rho\left[\partial_t X \partial_t^2 X + \cancel{v_s^2 \partial_x X\, \partial_x \partial_t X}\right]$$

$$\partial_x T^{xt} = -\rho v_s^2\left[\partial_x^2 X \partial_t X + \cancel{\partial_x X\, \partial_x \partial_t X}\right]$$

5 What is the consequence of space-translation symmetry?

Also: $x \to x + \varepsilon$ $\quad [Y^t = 0,\ Y^x = 1]$

$$J^t_{(ST)} = -\frac{\partial \mathcal{L}}{\partial(\partial_t X)}\partial_x X = -\rho\, \partial_t X\, \partial_x X = \underbrace{-T^{tx}}$$

$$J^x_{(ST)} = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_x X)}\partial_x X = \frac{\rho}{2}(\partial_t X)^2 + \frac{\rho}{2}v_s^2(\partial_x X)^2 = -T^{xx}$$

Again: $\partial_\mu J^\mu_{(ST)} = 0$

Recall: in equil.

$$X = x$$

actual loc. of displaced

pos. of block at rest

$$T^{tx} = \rho\, \partial_t X \underbrace{\partial_x X}_{\approx 1}$$

in/near equil.

$$T^{xx} = -\frac{\rho}{2}\left[(\partial_t X)^2 + v_s^2(\partial_x X)^2\right]$$

Linearized dynamics: $X = x + \underbrace{\delta X}_{\text{infinitesimal}}$:

$$T^{tx} = \rho\left[\partial_t \delta X + \cancel{\partial_t \delta X \cdot \partial_x \delta X}\right]; \quad T^{xx} \approx -\rho v_s^2 \partial_x \delta X$$