PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 15

The Klein-Gordon equation

September 26

1 Use effective field theory to deduce the minimal theory for a single relativistic field.

Goal: one field ϕ, $\underline{\mathcal{L}}$ invariant under trans. / Lorentz

Build $\mathcal{L}$ out of "invariant building blocks":

Ex 1: B.B. = ϕ $\qquad \phi(x^\mu) \to \phi(x^\mu + \varepsilon^\mu)$ isn't invariant?

But. $S = \int \underbrace{d^4x}_{dt\,dx\,dy\,dz} \mathcal{L}_1(\phi(x)) \longmapsto \int d^4\underbrace{(x+\varepsilon)}_{x'} \mathcal{L}[\phi(x+\varepsilon)] = \int d^4x' \mathcal{L}(\phi(x'))$

Ex. 2: B.B = $\partial_\mu\phi\,\partial^\mu\phi = \partial_\mu\phi\,\partial_\nu\phi\,\eta^{\mu\nu}$

[$\partial_\mu\phi \to \Lambda_\mu{}^\nu \partial_\nu\phi \neq \partial_\mu\phi$ is not B.B]

Our most general $\mathcal{L}$ [w/o 2-der. on ϕ]: $\mathcal{L}_2(\phi, \partial_\mu\phi\,\partial^\mu\phi)$

Suppose ϕ small ($\phi \simeq 0$). Taylor expand $\mathcal{L}_2$:

Klein-Gordon $\qquad \mathcal{L}_2 = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2$ $\qquad$ [ϕ-linear term dropped for stability]

2 What are the equations of motion for a Klein-Gordon field? Interpret them in terms of relativistic particle dynamics.

K-G theory: $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2$

EOMs: $0 = \frac{\delta S}{\delta \phi} = \underbrace{-m^2\phi}_{\frac{\partial \mathcal{L}}{\partial \phi}} + \underbrace{\partial_\mu \partial^\mu \phi}_{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}}$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial}{\partial(\partial_\mu\phi)}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi\right)$$
$$= -\frac{1}{2}\eta^{\alpha\beta}\left(\delta^\mu_\alpha\partial_\beta\phi + \delta^\mu_\beta\partial_\alpha\phi\right)$$
$$= -\partial^\mu\phi$$

For relativistic physics, work in units w/ $c=1$, ($\hbar = 1$)

$m^2\phi = \partial_\mu\partial^\mu\phi = -\partial_t^2\phi + \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi$

Plane wave solutions!

$\phi = e^{ikx - i\omega t}$

$m^2 \cancel{e^{\cdots}} = -(-i\omega)^2 \cancel{e^{\cdots}} + (ik)^2 \cancel{e^{\cdots}}$

$\boxed{\omega^2 = k^2 + m^2}$

$(mc^2)^2 + (pc)^2 = E^2$

QM: $\omega = E/\hbar$, $k = p/\hbar$

Most general:

$\phi = \int d^3k \; A(k)\, e^{ikx + i\sqrt{k^2+m^2}\,t} + B(k)\, e^{ikx - i\sqrt{k^2+m^2}\,t}$

Naive: ϕ as wave function?

Not quite.

3 What is the stress-energy tensor?

K-G theory: invariant under $x^\mu \to x^\mu + \varepsilon^\mu = x^\mu + \varepsilon^\nu \delta^\mu_\nu$

Goal: conserved stress-energy tensor $T^{\mu\nu}$. $(\partial_\mu T^{\mu\nu} = 0)$

Noether's Thm: if $x^\mu \to x^\mu + \varepsilon^\nu Y^\mu_\nu$ leaves S inv.

$(\partial_\mu J^\mu = 0)$ $\quad J^\mu_\nu = \mathcal{L} Y^\mu_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi Y^\rho_\nu$

Translations: $Y^\mu_\nu = \delta^\mu_\nu$!

plug in K-G $\mathcal{L}$

$$T^\mu{}_\nu = \mathcal{L} \delta^\mu{}_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi \, \delta^\rho_\nu = \partial^\mu \phi \partial_\nu \phi - \delta^\mu_\nu \frac{1}{2}(m^2\phi^2 + \partial_\rho \phi \partial^\rho \phi)$$

Use η to raise ν:

$$\eta^{\nu\rho} T^\mu{}_\nu = \boxed{T^{\mu\rho} = \partial^\mu \phi \partial^\rho \phi - \frac{1}{2}\eta^{\mu\rho}(m^2\phi^2 + \partial_\sigma \phi \partial^\sigma \phi)}$$

$\boxed{4}$ Why is $T^{\mu\nu} = T^{\nu\mu}$?

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(m^2\phi^2 + \partial_\sigma\phi\partial^\sigma\phi) = T^{\nu\mu}$$

$\underset{\substack{\uparrow \\ \text{spatial index}}}{T^{ti}} = T^{it}$. up to factors of c: (momentum dens.)$_i$ = i-comp. of energy cur.

Why? Noether's Thm applied to Lorentz!

$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon^\mu{}_\nu$

$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}$

Plug in $\Lambda = \delta + \varepsilon$:

$\cancel{\eta^{\mu\nu}} = \cancel{\delta^\mu{}_\rho \delta^\nu{}_\sigma \eta^{\rho\sigma}} + \varepsilon^\mu{}_\rho \eta^{\rho\nu} + \varepsilon^\nu{}_\sigma \eta^{\mu\sigma}$

$0 = \varepsilon^{\mu\nu} + \varepsilon^{\nu\mu}$

$x^\mu \to x^\mu + \varepsilon^{\mu\nu} x_\nu$

$\varepsilon^{\mu\nu} x_\nu = \underbrace{\frac{1}{2}\delta^\mu_\rho}_{\text{trans?}} [\varepsilon^{\rho\nu} x_\nu - \varepsilon^{\nu\rho} x_\nu]$

$\downarrow$ conserved current:

$J^\mu(\varepsilon) = \frac{1}{2} T^\mu{}_\rho [\varepsilon^{\rho\nu} x_\nu - \varepsilon^{\nu\rho} x_\nu] = J^\mu{}_{\rho\nu} \varepsilon^{\nu\rho}$

where $\partial_\mu J^\mu{}_{\nu\rho} = 0$

$\partial_\mu J^\mu{}_{\rho\nu} = \partial_\mu \frac{1}{2}(\underbrace{T^\mu{}_\rho x_\nu} - T^\mu{}_\nu x_\rho)$

$\cancel{(\partial_\mu T^\mu{}_\rho)}^{\to 0} x_\nu + \underbrace{T^\mu{}_\rho \partial_\mu x_\nu}_{T^\mu{}_\rho \eta_{\mu\nu} = T_{\nu\rho}}$

$= T_{\nu\rho} - T_{\rho\nu} = 0$.