PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 16

Relativistic electromagnetism

September 28

1 Discuss gauge symmetry of electromagnetism.

= redundancy in description

In elec-km: $A_\mu \to A_\mu + \partial_\mu \lambda$ ← arbitrary functions

FIELDS invariant b/c $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = $

$$\begin{array}{cccc} t & x & y & z \end{array}$$
$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

in units where $c=1$

$\left(E_i \to \frac{E_i}{c}\right)$

better to write $\mathcal{L}(A_\mu, \partial_\nu A_\mu, \ldots)$

Perspective 1: $A_\mu \to A_\mu + \partial_\mu \lambda$ "big symmetry", write $\mathcal{L}$ out of invariant B.B.s.

Perspective 2: $A_\mu \sim A_\mu + \partial_\mu \lambda$

same config in Hilbert space!

2 Write down the minimal gauge-invariant Lagrangian.

$$\mathcal{L}(A_\mu, \underline{\partial_\nu A_\mu}, \ldots) = \mathcal{L}(A_\mu + \partial_\mu\lambda, \partial_\nu A_\mu + \partial_\nu\partial_\mu\lambda, \ldots) \quad (\cancel{+\partial_\mu K^\mu})$$

Build $\mathcal{L}$ out of $\underbrace{\text{invariants}}$ under λ-trans:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \mapsto \partial_\mu A_\nu + \cancel{\partial_\mu\partial_\nu\lambda} - \partial_\nu A_\mu - \cancel{\partial_\nu\partial_\mu\lambda} = F_{\mu\nu}.$$

field

Minimal Lagrangian: $\mathcal{L}(A) = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -\frac{1}{4}\eta^{\alpha\gamma}\eta^{\beta\delta}\left[\delta^\nu_\alpha\delta^\mu_\beta - \delta^\nu_\beta\delta^\mu_\alpha\right](\partial_\gamma A_\delta - \partial_\delta A_\gamma)\cdot 2$$

$$= -\frac{1}{2}\eta^{\nu\gamma}\eta^{\mu\delta}F_{\gamma\delta} + \ldots$$

$$= \frac{F^{\mu\nu}}{2} - \frac{F^{\nu\mu}}{2} = F^{\mu\nu}$$

EOMs:

$$0 = \frac{\delta S}{\delta A_\mu} = \cancel{\frac{\partial \mathcal{L}}{\partial A_\mu}} - \partial_\nu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}}$$

$$\mathcal{L} = -\frac{1}{4}\eta^{\alpha\gamma}\eta^{\beta\delta}\underbrace{(\partial_\alpha A_\beta - \partial_\beta A_\alpha)}(\partial_\gamma A_\delta - \partial_\delta A_\gamma)$$

free space Maxwell eqs ($J^\mu = 0$)

$$\boxed{\partial_\mu F^{\mu\nu} = 0}$$

3 Discuss the solutions to the equations of motion.

Found: $\partial_\mu F^{\mu\nu} = 0 = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$

Plane wave: $A_\mu = a_\mu e^{i \underbrace{k_\alpha x^\alpha}} = a_\mu e^{i[k_x x + k_y y + k_z z - \omega t]}$

a_μ constant.

$k^\mu = \begin{pmatrix} \omega \\ k_x \\ k_y \\ k_z \end{pmatrix}$

$\partial^\mu F_{\mu\nu} = 0$: $\partial_\mu \to i k_\mu$ and $\partial^\mu = i k^\mu$

find: $k^\mu [k_\mu a_\nu - k_\nu a_\mu) = 0 = (k^\mu k_\mu) a_\nu - k_\nu \cancel{(k^\mu a_\mu)}$

Claim: use gauge invariance to set $k^\mu a_\mu = 0$.

$0 = (k^\mu k_\mu) \cancel{(a_\nu)}$

$0 = -\omega^2 + (k_x^2 + k_y^2 + k_z^2) c^2$

$\underbrace{\omega = c|\vec{k}|}$

EM waves travel w/ speed c

4 Why are there only 2 allowable polarizations?

Before: ① $\underbrace{k^\mu k_\mu = 0}$ and ② $\underbrace{k^\mu a_\mu = 0}$

If $c_1 \neq 0$:

$A^\mu = \partial^\mu \lambda + 0$

$\lambda = \frac{c_1}{i} e^{ik^\mu x_\mu}$

$a^\mu = c_1 k^\mu$

[For plane wave:

$A^\mu = a^\mu e^{ik^\alpha x_\alpha}$

Assume: $k^\mu = \begin{pmatrix} k \\ k \\ 0 \\ 0 \end{pmatrix}$

What terms allowed in a?

$\underbrace{\begin{pmatrix} k \\ k \\ 0 \\ 0 \end{pmatrix}}_{k^\mu} \underbrace{\begin{pmatrix} -a^t \\ a^x \\ a^y \\ a^z \end{pmatrix}}_{a_\mu} = 0 = k(-a^t + a^x)$

$a^t = a^x$

a^y, a^z unconst.

What if $k^\mu a_\mu \neq 0$? $[k^\mu k_\mu \neq 0]$

Try 1: $k^\mu = \begin{pmatrix} \omega \\ 0 \\ 0 \\ 0 \end{pmatrix}$:

$\rightarrow \begin{cases} -\omega^2 a_\nu = 0 & (\nu = x, y, z) \\ -\omega^2 a_t + \omega^2 a_t = 0 & (\nu = t) \end{cases}$

only gauge sol'n

Most general solution:

$a^\mu = \begin{pmatrix} c_1 k \\ c_1 k \\ c_2 \\ c_3 \end{pmatrix} = c_1 k^\mu + \boxed{\begin{pmatrix} 0 \\ 0 \\ c_2 \\ c_3 \end{pmatrix}}$

$c_{1,2,3}$ arbitrary const.

2 polarizations for EM waves!

5 Find the stress tensor $T^{\mu\nu}$.

Last time: Lorentz-invariance implies $T^{\mu\nu} = T^{\nu\mu}$.

$$T^{\mu\nu} = \mathcal{L}\,\eta^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho$$

$$= -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} - F^{\rho\mu}\partial^\nu A_\rho$$

Does $T^{\mu\nu} = T^{\nu\mu}$? NO!

$$F^{\rho\mu}\partial^\nu A_\rho - F^{\rho\nu}\partial^\mu A_\rho = (\partial^\rho A^\mu - \cancel{\partial^\mu A^\rho})\partial^\nu A_\rho - (\partial^\rho A^\nu - \cancel{\partial^\nu A^\rho})\partial^\mu A_\rho$$

If we take $T^{\mu\nu}_{sym} = T^{\mu\nu} + \underbrace{F^{\rho\mu}\partial_\rho A^\nu}$

$$= -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\eta^{\mu\nu} - F^{\rho\mu}F^{\nu}{}_{\rho}$$

symmetric!

Why??

$$F^{\rho\mu}\partial_\rho A^\nu = \underbrace{\partial_\rho(F^{\rho\mu}A^\nu)}_{\text{total divergence.}} - \underbrace{A^\nu\partial_\rho F^{\rho\mu}}_{=0 \text{ on EOMs.}}$$

$$\int d^4x\, T^{\mu\nu} = \int d^4x\, T^{\mu\nu}_{sym} + \underbrace{0}_{\text{boundary}}$$