PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 17

Coupling electromagnetism and matter

September 30

1 Discuss the complex Klein-Gordon theory.

"one" field ϕ : complex-valued: $\phi_1 + i\phi_2 = \phi$

$\phi_1 - i\phi_2 = \bar{\phi}$ $(=\phi^*)$

$\phi, \bar{\phi}$ independent

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \bar{\phi}\, \partial^\mu \phi - \frac{1}{2} m^2 \underbrace{\bar{\phi}\phi}_{=\phi_1^2 + \phi_2^2}$$

$$\underbrace{-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \cdots}$$

Equations of motion:

$$\frac{\delta S}{\delta \phi} = 0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

$$= -\frac{1}{2} m^2 \bar{\phi} - \partial_\mu \left(-\frac{1}{2} \partial^\mu \bar{\phi} \right)$$

$$m^2 \bar{\phi} = \partial_\mu \partial^\mu \bar{\phi}$$

Same solutions: $\phi = \int dk\, A(k) e^{ikx - i\omega t}$

$\omega^2 = k^2 + m^2$

$\frac{\delta S}{\delta \bar{\phi}} = 0$ ... similar

$$\boxed{m^2 \phi = \partial_\mu \partial^\mu \phi}$$ same as K-G theory

2 What is the conserved current associated to phase rotation?

Complex K-G has $\underbrace{U(1)\text{ symmetry}}$: $\phi \to \phi e^{i\lambda}$, $\bar\phi \to \bar\phi e^{-i\lambda}$ c.c.

$e^{-i\lambda + i\lambda} = 1$

$$\mathcal{L} \to -\frac{1}{2}\partial_\mu(\phi \cancel{e^{i\lambda}})\,\partial^\mu(\bar\phi \cancel{e^{-i\lambda}}) - \frac{1}{2}m^2(\bar\phi \cancel{e^{-i\lambda}})(\phi \cancel{e^{i\lambda}})$$

$U(1) = 1\times 1$ $\underbrace{\text{unitary}}_{U^\dagger = U^{-1}}$ matrices : complex # u, $u^* = \bar u = \frac{1}{u}$ } solved by; $u = e^{i\lambda}$

$\lambda \to$ infinitesimal ε

Noether's Thm: $\phi \to \phi + \underbrace{i\varepsilon\phi}$, $\bar\phi \to \bar\phi - \underbrace{i\varepsilon\bar\phi}$

$g_\phi = i\phi$ $\qquad g_{\bar\phi} = -i\bar\phi$

conserved current J^μ

$$J^\mu = i\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - i\bar\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\phi)} = \frac{i}{2}\left[-\phi\partial^\mu\bar\phi + \bar\phi\partial^\mu\phi\right]$$

$\boxed{3}$ Promote this symmetry to a "gauge symmetry".

Global symmetry: $\phi(x^\mu) \to \phi(x^\mu) e^{i\lambda}$, λ x-independent.

$\downarrow$ Local ???

$\phi(x) \to \phi(x) e^{i\lambda(x)}$. "Gauge symmetry"

$\begin{cases} \phi \to \phi e^{i\lambda} \\ \bar\phi \to \bar\phi e^{-i\lambda} \end{cases}$

Building blocks dependent on $\phi, \bar\phi$?

$(\partial_\mu(\phi\bar\phi)$ OK$)$ $\phi\bar\phi$ is invariant

Include derivatives on ϕ: $\partial_\mu(\phi e^{i\lambda}) \to (\partial_\mu\phi + \phi i \partial_\mu\lambda) e^{i\lambda}$

Salvage $\partial_\mu\phi$... we add $A_\mu \to A_\mu + \partial_\mu\lambda$

$$D_\mu\phi = \partial_\mu\phi - iA_\mu\phi \longmapsto (\partial_\mu\phi + \cancel{\phi i \partial_\mu\lambda}) e^{i\lambda} - i(A_\mu + \cancel{\partial_\mu\lambda}) e^{i\lambda}\phi$$
$$\longmapsto (D_\mu\phi) e^{i\lambda(x)}$$

New B.B: $D_\mu\phi D^\mu\bar\phi$

$D_\mu\bar\phi = \partial_\mu\bar\phi + iA_\mu\bar\phi$

4 Couple the theory to dynamical electromagnetism.

Write $\mathcal{L}(\phi, \bar{\phi}, A_\mu)$ gauge-invariant.

Building blocks: $\phi\bar{\phi}$, $D_\mu \phi D^\mu \bar{\phi}$, $F_{\mu\nu}F^{\mu\nu}$

effective field theory:

$$\mathcal{L} = -\frac{1}{2} D_\mu \phi D^\mu \bar{\phi} - \frac{m^2}{2}\bar{\phi}\phi - \frac{1}{4}\underbrace{F_{\mu\nu}F^{\mu\nu}}_{\text{gives } A_\mu \text{ dynamics}}$$

[constant prefactors convenient].

5 What are the equations of motion?

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi - iA_\mu \phi)(\partial^\mu \bar{\phi} + \underbrace{iA^\mu \bar{\phi}}) - \frac{m^2}{2}\phi\bar{\phi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\frac{\delta S}{\delta \bar{\phi}} = \frac{\partial \mathcal{L}}{\partial \bar{\phi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} = 0$$

$$= -\frac{m^2}{2}\phi - \frac{i}{2}A^\mu(\partial_\mu \phi - iA_\mu \phi) - \partial^\mu\left(-\frac{1}{2}(\partial_\mu \phi - iA_\mu \phi)\right)$$

$$0 = -m^2\phi + \underbrace{(\vec{\partial}^\mu - iA^\mu)(\vec{\partial}_\mu - iA_\mu)\phi}_{\partial^\mu \partial_\mu \phi - 2iA^\mu \partial_\mu \phi - i(\partial_\mu A^\mu)\phi - A_\mu A^\mu \phi}$$

intrinsically nonlinear due to A.

$$\frac{\delta S}{\delta A_\mu} = \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu F^{\nu\mu}$$

$$\frac{i}{2}\phi\partial^\mu \bar{\phi} - \frac{A^\mu \bar{\phi}\phi}{2} - i\bar{\phi}\partial^\mu \phi - \frac{A^\mu \phi\bar{\phi}}{2}$$

$$= -\partial_\nu F^{\nu\mu} + \frac{i}{2}\phi(\partial^\mu \bar{\phi} + iA^\mu \bar{\phi}) - \frac{i}{2}\bar{\phi}(\partial^\mu \phi - iA^\mu \bar{\phi})$$

$$\partial_\nu F^{\nu\mu} = \frac{i}{2}(\phi D^\mu \bar{\phi} - \bar{\phi} D^\mu \phi) \leftarrow J^\mu$$