PHYS 5210

Graduate Classical Mechanics

Fall 2022

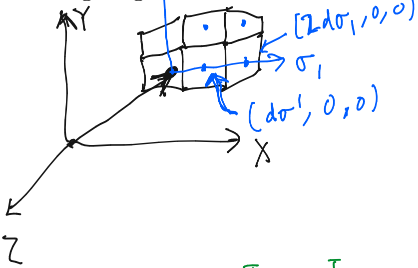
Lecture 19

The elastic stress tensor

October 5

1 Review the Lagrangian for a solid.

Last time:



Lagrangian for $\sigma^I(X_i, t)$

+ higher deriv.

$$\mathcal{L} = A\,\partial_t\sigma^I\partial_t\sigma^I - B\,\partial_i\sigma^I\partial_i\sigma^I - C\,\partial_i\sigma^I\partial_j\sigma^I\partial_i\sigma^J\partial_j\sigma^J - \dots$$

- t-trans; X_i, σ^I trans.
- $SO(3)$ for X_i or $\boxed{\sigma^I}$

HW8 (?) broken sym. for crystal

Equilibrium: $\underline{\sigma^I = X_i\delta_i^I}$

$\sigma^I = X_i\delta_i^I + \phi^I$

small perturb.

2 Find the stress tensor associated with X_i translation symmetry.

missing minus sign

$$-T^{ji} = -\mathcal{L}\,\delta^{ji} + \frac{\partial \mathcal{L}}{\partial(\partial_j \sigma^K)}\partial_i \sigma^K = -\delta^{ji}\left[\cancel{A\partial_t\sigma^I\partial_t\sigma^I} - B\partial_k\sigma^I\partial_k\sigma^I - C\partial_k\sigma^I\partial_k\sigma^J\partial_l\sigma^I\partial_l\sigma^J\right] - \left[2B\partial_j\sigma^K + 4C\partial_j\sigma^L\partial_l\sigma^L\partial_l\sigma^K\right]\partial_i\sigma^K$$

flux of i-momentum in X_j-direction.

Expand $\sigma^I = X_i\delta_i^I + \phi^I$:

$\delta_k^I \quad \delta^{IJ}$

$$T^{ji} = \delta^{ji}\left[2B\delta_k^I\partial_k\phi^I + 4C\partial_k\phi^I\delta_k^J\delta_l^I\delta_l^J\right]$$
$$-2B(\partial^j\phi^K\delta^{iK} + \delta^{jK}\partial^i\phi^K) - 4C(\partial^j\phi^L\delta_i^K + \partial^i\phi^K\delta_j^L)\delta_l^L\delta_l^K$$
$$-4C\delta_j^L\delta_i^K(\partial_l\phi^L\delta_l^K + \partial_l\phi^K\delta_l^L)$$

$$\boxed{= \delta^{ji}(2B+4C)\partial_l\phi^l - (\partial^i\phi^j + \partial^j\phi^i)(2B+8C)}$$

3 What are the constraints on T_{ji} due to stability of a solid?

Write: $$\underbrace{\partial_i\phi_j + \partial_j\phi_i}_{2u_{ij}} = \underbrace{\partial_i\phi_j + \partial_j\phi_i - \frac{2}{3}\delta_{ij}\partial_k\phi_k}_{\text{traceless}} + \underbrace{\frac{2}{3}\delta_{ij}\partial_k\phi_k}_{\text{trace}}$$

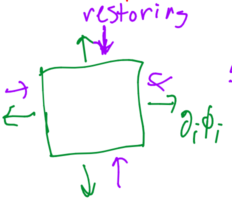
(strain)

$$-T_{ij} = -\mu\left[\partial_i\phi_j + \partial_j\phi_i - \frac{2}{3}\delta_{ij}\partial_k\phi_k\right] - K\delta_{ij}\partial_k\phi_k$$

$\mu = 2B + 8C$ $\qquad$ $\underline{K = \frac{4C}{3} - \frac{2B}{3}}$.

Claim: $\mu > 0$ and $K > 0$.

Stability of $\sigma^I = X_i\delta_i^I$ require $\underline{\mu > 0 \text{ and } K > 0}$.



$\underline{\Delta P < 0}$

$T_{ij} = P\delta_{ij}$

4 Discuss the conventional formulation of elasticity in terms of a stress/strain tensor.

mistake in class!

$$-u_{ij} = \frac{1}{2}(\partial_i \phi_j + \partial_j \phi_i) \quad : \quad \text{Strain tensor}$$

in general: $T_{ij} = -\underbrace{\lambda_{ijkl}}_{\text{elastic moduli / Lamé coeff.}} u_{kl}$

For isotropic solid:

$$\lambda_{ijkl} = \underbrace{K\delta_{ij}\delta_{kl}}_{\text{(spin 0)}} + \underbrace{2\mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)}_{\text{(spin 2)}}$$

Throwback: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

$$\cancel{\frac{1}{2}\left[\partial_i \phi_j - \partial_j \phi_i\right]}$$

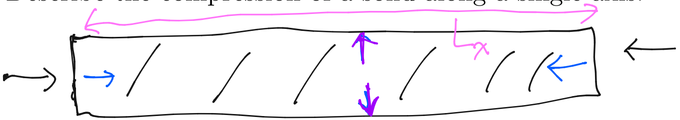
$\phi_y = +\varepsilon x$

$\phi_x = -\varepsilon y$

rotation!

5 Describe the compression of a solid along a single axis.

"uniaxial"



external T_{xx}.

$$u_{ij} = \frac{1}{2}(\partial_i \phi_j + \partial_j \phi_i)$$

$$u_{xx} = \frac{\Delta L_x}{L_x}$$

$$T_{ij} = -K u_{kk} \delta_{ij} - 2\mu\left(u_{ij} - \frac{1}{3}\delta_{ij} u_{kk}\right)$$

$$T_{ii} = -3K u_{ii}$$

$$u_{ij} = -\frac{1}{2\mu}\left(T_{ij} - \frac{1}{3}T_{kk}\delta_{ij}\right) - \frac{T_{kk}}{9K}\delta_{ij}$$

Compress in x-direction:

$$u_{xx} = \partial_x \phi_x = -\frac{1}{2\mu}\cdot\left(1 - \frac{1}{3}\right)T_{xx} - \frac{T_{xx}}{9K} = -\frac{T_{xx}}{Y}$$

Y is Young's modulus: $Y = \frac{9K\mu}{\mu + 3K} \sim 10^9 - 10^{11}$ Pa.

$$u_{yy} = \frac{1}{2\cdot 3\mu} T_{xx} - \frac{T_{xx}}{9K}$$