PHYS 5210

Graduate Classical Mechanics

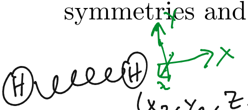
Fall 2022

Lecture 2

Harmonic oscillation

August 24

1 Describe the (classical) vibration of a diatomic molecule. What are the symmetries and principles of nature we should use to constrain S?



(x_2, y_2, z_2)

(x_1, y_1, z_1)

"ignore" center of mass motion:

$(\Delta x, \Delta y, \Delta z)$

$x_1 - x_2$

rotational invariance:

only Δx of interest

bond length

Assume locality in time

... exhausted exact symmetries

$$S = \int dt\ L(\Delta x, \Delta \dot{x}, \Delta \ddot{x}, \ldots)$$

Focus on a useful limit:

slow dynamics, near equilibrium

$\Delta \ddot{x} \ll \Delta \dot{x}$

ignore all derivatives except $\dot{q}$ in L

at rest: $\Delta x = D_0$

$\Delta x = D_0 + q$

$\frac{|q|}{D_0} \ll 1$

$$\boxed{S = \int dt\ L(q, \dot{q})}$$

2 Use effective theory arguments to find S for weakly perturbed motion.

q is <u>small</u> : Taylor expand L in $q, \dot{q}$:

$$L = A + Bq + Cq^2 + F\dot{q} + Gq\dot{q} + H\dot{q}^2 + \cdots$$

$$= \frac{d}{dt}\left[Fq + \frac{1}{2}Gq^2\right]$$

Set $A = 0$!

$$S = \int_{t_i}^{t_f} dt\, L$$

$$S = \underbrace{A(t_f - t_i)}_{\text{nothing to vary!}} + \cdots$$

(useful)

Set $B = 0$.

If $B \neq 0$, then as $q \to 0$,

$L \approx Bq$

S minimized by $q < 0$.

But equilib. is $q = 0$.

(physics)

$$S = \int dt\, L$$

$$= \cdots + \left[Fq + \frac{1}{2}Gq^2\right]\Big|_{t_i}^{t_f}$$

But $q(t_f)$ and $q(t_i)$ are fixed!

can't vary over trajectories!

(useful)

Hence

$$L = Cq^2 + H\dot{q}^2$$

3 Describe the resulting dynamics of a harmonic oscillator.

Eff Theory:

Generic phys. system near equil. described by this theory.

$$L = Cq^2 + H\dot{q}^2$$

$$0 = \frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

$$0 = \cancel{2}Cq - \frac{d}{dt}(\cancel{2}H\dot{q})$$

$$\underline{H\ddot{q} = Cq}$$

If $\frac{C}{H} > 0$, unstable

[not stable, bound...]

Hence $\frac{C}{H} < 0$

Solve this ODE:

Try $q = e^{rt}$:

$$\cancel{e^{rt}} Hr^2 = C\cancel{e^{rt}}$$

$$r = \pm\sqrt{\frac{C}{H}} = \pm r_0$$

$$q(t) = C_1 e^{r_0 t} + C_2 e^{-r_0 t}$$

const. of integration.

$H = \frac{1}{2}m$ (m mass)

$C = -\frac{1}{2}m\omega_0^2$: $r_0 = i\omega_0$

$$q = q_1 e^{i\omega_0 t} + q_2 e^{-i\omega_0 t}$$

$$= q_1 \cos(\omega_0 t) + q_2 \sin(\omega_0 t)$$

4 Give a careful scaling argument which justifies neglecting corrections to our effective theory of the harmonic oscillator.

Self-consistent? [mandates $q_H < 0 \ldots$]

1) $q^3, q^4, \ldots$ negligible in L?

$$S = \int dt \left[\frac{m}{2}\dot{q}^2 - \frac{m}{2}\omega_0^2 q^2 + \alpha q^3 + \ldots\right]$$

$q = \lambda \tilde{q}$:

$$S = \int dt\, \lambda^2 \left[\frac{m}{2}\dot{\tilde{q}}^2 - \frac{m}{2}\omega_0^2 \tilde{q}^2 + \lambda \cdot \alpha \tilde{q}^3\right]$$

In limit $\lambda \to 0$, α ignored.

α is "irrelevant" term

In physical problems:
E.T. accurate until $q \sim D_0$. Here,
$m\omega_0^2 D_0^2 \sim \alpha D_0^3$. Suggest $\alpha \sim \dfrac{m\omega_0^2}{D_0}$

2) $\ddot{q}$ negligible vs. $\dot{q}$?

Heuristic: $\dfrac{\ddot{q}}{\dot{q}} \sim \dfrac{1}{\text{time scale of dyn}}$

$\sim \omega_0$

Formally: if we drop $\ddot{q}$, assumption that:

$$L = L_{\text{eff}} + J\ddot{q}^2,$$

assuming $J\omega_0^2 \ll m$.

Practical: J doesn't qualitatively change, so ignore

5 Generalize the harmonic oscillator to higher dimensional problems, assuming time-reversal symmetry.

If system is described by $(q_1, \dots q_N)$

Stable Equilibrium is $q_i = 0$ [implicit for all i]

Time-reversal $t \to -t$, $\dot{q}_i \to -\dot{q}_i$

If L is invariant under TR

Taylor expand: $L = \frac{1}{2} \dot{q}_i M_{ij} \dot{q}_j - \frac{1}{2} \underbrace{q_i K_{ij} q_j}_{} + \cdots$

Einstein sum. convention

$$q_i K_{ij} q_j = \sum_{i=1}^{N} \sum_{j=1}^{N} q_i K_{ij} q_j$$