PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 20

Sound waves in solids

October 7

1 Review the Lagrangian and stress tensor for an elastic solid.

$$\mathcal{L} = A\partial_t\sigma^I\partial_t\sigma^I - B\partial_j\sigma^I\partial_j\sigma^I - C\partial_i\sigma^K\partial_j\sigma^K\partial_i\sigma^L\partial_j\sigma^L + \dots$$

Conserved momentum! Invariance under $X_i \to X_i + \varepsilon_i$:

$$\rightarrow \underline{\partial_t T^{ti} + \underbrace{\partial_j T^{ji}} = 0}$$

raised/lower index OK ✓

if $\sigma^I = \underline{X_i\delta^I_i} + \phi^I$

mom. dens.

Lec 19: $-T_{ji} = -\mu(\partial_i\phi_j + \partial_j\phi_i - \frac{2}{3}\delta_{ij}\partial_k\phi_k) - K\delta_{ij}\partial_k\phi_k$

$\mu = 2B + 8C$, $K = \frac{4C}{3} - \frac{2B}{3}$.

mistake in class, missing minus signs!

$$-T^{ti} = \frac{\partial\mathcal{L}}{\partial(\partial_t\sigma^I)}\partial_i\sigma^I = (2A\partial_t\sigma^I)\partial_i\sigma^I \approx 2A\partial_t\phi^I\cancel{\partial_i X^I}\delta^I_i$$

$$= 2A\partial_t\phi_i$$

$i = x, y, z$.

$$2A\partial_t^2\phi_i - \partial_j\left[\mu\partial_j\phi_i + \mu\partial_i\phi_j + \left(K - \frac{2\mu}{3}\right)\delta_{ij}\partial_k\phi_k\right] = 0.$$

$$\partial_i\partial_j\phi_j$$

$$\parallel$$

$$\partial_j\delta_{ij}\partial_{\underline{k}}\phi_{\underline{k}} = \partial_i\partial_{\underline{k}}\phi_{\underline{k}}$$

$$\underset{\sim}{2A}\,\partial_t^2\phi_i = \mu(\partial_j\partial_j)\phi_i + \left(K + \frac{\mu}{3}\right)\partial_i\partial_j\phi_j$$

ρ = mass density

$$-\rho\,\partial_t^2\phi_i = \frac{m a_i}{V} = \frac{F_i^{\text{elastic}}}{V}$$

2 Find the Euler-Lagrange equations.

Euler-L eqns: $$0 = \cancel{\frac{\partial \mathcal{L}}{\partial \sigma^I}} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \sigma^I)} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j \sigma^I)}$$

$$0 = -2A\,\partial_t^2\sigma^I + 2B\,\partial_j\partial_j\sigma^I + 4C\,\partial_\ell\left[\partial_\ell\sigma^J\,\partial_k\sigma^J\,\partial_k\sigma^I\right]$$

$\sigma^I = X_i\delta_i^I + \phi^I$: only keep linear terms in ϕ:

δ_i^I $\quad$ δ_k^J $\quad$ δ_k^I $\quad$ δ^{IJ} $\quad$ $\delta_\ell^J\delta_k^J\partial_k\phi^I$

$$2A\,\partial_t^2\phi_i = 2B\,\partial_j\partial_j\phi_i + 4C\,\partial_\ell\left[\partial_\ell\phi^J\,\cancel{\partial_k X^J}\,\cancel{\partial_k X^I} + \delta_\ell^J\,\partial_k\phi^J\,\delta_k^I + \underline{\delta_\ell^J\delta_k^J\partial_k\phi^I}\right]$$

$$+4C\,\partial_\ell\partial_\ell\phi_i + 4C\,\partial_i\partial_\ell\phi_\ell + 4C\,\partial_\ell\partial_\ell\phi_i$$

$$\underbrace{2A}_{\rho}\,\partial_t^2\phi_i = \underbrace{(2B+8C)}_{\mu}\,\partial_j\partial_j\phi_i + \underbrace{4C}\,\partial_i\partial_j\phi_j$$

$$K + \frac{\mu}{3} = \frac{4C}{3} - \frac{\cancel{2B}}{3} + \frac{\cancel{2B}+8C}{3} = 4C \ \checkmark$$

3 Find the plane wave solutions.

$$\rho\, \partial_t^2 \phi_i = \mu\, \partial_j \partial_j \phi_i + \left(K + \frac{\mu}{3}\right) \partial_i \partial_j \phi_j$$

Look for soln: $\phi_j = a_j e^{ik_\ell x_\ell - i\omega t}$

a_j ← const. vector

$\partial_t \to -i\omega \qquad \partial_j \to i k_j$

$$M_{ij} = \underline{\mu k_\ell k_\ell \delta_{ij}} + \underline{\left(K + \frac{\mu}{3}\right) k_i k_j}$$

$$-\rho\omega^2 a_i = -\mu k_j k_j a_i - \left(K + \frac{\mu}{3}\right) k_i k_j a_j = -M_{ij}(k)\, a_j$$

Find e-values/e-vectors of M. Use rotational symmetry to $\underline{\text{choose } \vec{k} = (0,0,k)}$.

$$M = \begin{pmatrix} \boxed{\mu k^2} & 0 & 0 \\ 0 & \boxed{\mu k^2} & 0 \\ 0 & 0 & \boxed{\mu k^2 + \left(K + \frac{\mu}{3}\right) k^2} \end{pmatrix}$$

S-waves:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{ik(z - v_s t)} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{ik(z - v_s t)}$$

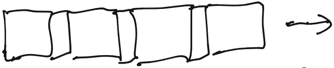
P-wave:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{ik(z - v_p t)}$$

$$\omega^2 = \frac{\mu}{\rho} k^2 = v_s^2 k^2 \qquad \omega^2 = \frac{K + \frac{4}{3}\mu}{\rho} k^2 = v_p^2 k^2$$

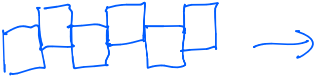
4 Discuss the two types of sound waves in a solid.

P-wave:



longitudinal (compression) sound wave

S-wave:



transverse (shear) sound wave.

$$v_p = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}$$

$$v_s = \sqrt{\frac{\mu}{\rho}}$$

$K, \mu \sim 10^9 - 10^{11}$ Pa

$\rho \sim \cancel{3\times} 10^3$ kg/m^3

Estimate:

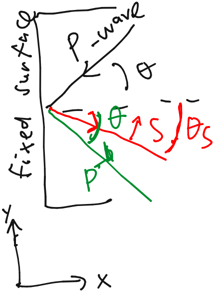
$v_s, v_p \sim 10^3 - 10^4 \ \frac{m}{s}$.

$K, \mu > 0$

$$\frac{v_p}{v_s} = \sqrt{\frac{4}{3} + \frac{K}{\mu}} > \sqrt{\frac{4}{3}}.$$

$\underline{v_p > v_s}$

5 How does a P-wave reflect off a fixed/clamped boundary?



$$\vec{\phi}_{inc} = \vec{a}\, e^{i\vec{k}\cdot\vec{x} - \underbrace{i v_p k t}_{-i\omega t}}$$

$$\vec{k}\cdot\vec{x} = -k\cos\theta\, x - k\sin\theta\, y$$

$$\vec{a} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}.$$

Fixed boundary cond:

at $x=0$: $\phi_x = \phi_y = \phi_z = 0$.

$$\vec{\phi}_{p,ref} = \vec{\alpha}\, e^{i\vec{k}\cdot\vec{x} - i v_p k t} = \alpha \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} e^{ik[x\cos\theta - y\sin\theta - \omega t]}$$

$$\vec{\phi}_{s,ref} = \beta \begin{pmatrix} \sin\theta_s \\ \cos\theta_s \\ 0 \end{pmatrix} e^{ik_s[\cos\theta_s \cdot x - \sin\theta_s \cdot y - v_s t]}$$

Solve for k_s, θ_s:

$$v_p k = v_s k_s, \quad k_s = \frac{v_p}{v_s} k$$

$$\sin\theta_s = \sin\theta \frac{k}{k_s} = \sin\theta \frac{v_s}{v_p}$$

Solve α, β:

$$-\cos\theta + \alpha\cos\theta + \beta\sin\theta_s = 0$$

$$-\sin\theta - \alpha\sin\theta + \beta\cos\theta_s = 0$$