PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 21

Effective field theory of an ideal fluid

October 10

1 What are the symmetries of a fluid?

difference btwn fluid (gas/liquid) and solid is indifference to shear (volume-preserving transformation).

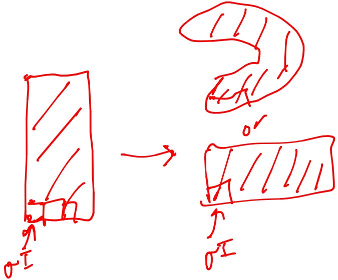
start w/ the same fields: $\sigma^I(X_i, t)$

(internal) fluid cell

physical coordinate

All symmetries of the solid remain:

- $t \to t + \epsilon_t$
- $X_i \to X_i + \epsilon_i$
- $X_i \to X_i + \epsilon_{ij} X_j$
  $\epsilon_{ij} = -\epsilon_{ji}$
- $\sigma^I \to \sigma^I + c^I$
- $\sigma^I \to \sigma^I + \epsilon^{IJ} \sigma^J$



volume-preserving coord transform:

$$\sigma^I(X) \to \xi^I(\sigma(X)) \quad \text{s.t.} \quad \det\left(\frac{\partial \xi^I}{\partial \sigma^J}\right) = 1$$

$$\sigma^I \to \xi^I(\sigma)$$

[B/c: $\int d^3\xi = \int \det\left(\frac{\partial \xi^I}{\partial \sigma^J}\right) d^3\sigma$]

2 What are the invariant building blocks?

Invariance under $\sigma^I \to \sigma^I + \xi^I$: $\underline{\partial_t \sigma^I}$, $\partial_j \sigma^I$ only

How to find invariants under $\underline{\sigma^I \to \xi^I(\sigma)}$, $\det\left(\frac{\partial \xi^I}{\partial \sigma^J}\right) = 1$?

$$\partial_t \sigma^I \to \partial_t \xi^I(\sigma) \xrightarrow{\text{chain rule}} \frac{\partial \xi^I}{\partial \sigma^J} \partial_t \sigma^J$$

$$M^{IJ} = \frac{\partial \xi^I}{\partial \sigma^J}$$

$$\partial_t \sigma^I \to M^{IJ} \partial_t \sigma^J$$

$\det(AB) = \det(A)\det(B)$

$\det(M) = \det(M^T)$

$$\det(\overbrace{\partial_j \sigma^I \partial_j \sigma^J}^{N^{IJ}}) \to \det\left(\frac{\partial \xi^I}{\partial \sigma^K} \partial_j \sigma^K \cdot \frac{\partial \xi^J}{\partial \sigma^L} \partial_j \sigma^L\right)$$

$\uparrow$ F

$$= \det\left(M^{IK} N^{KL} (M^T)^{LJ}\right) = \det(M N M^T)$$

$$= \cancel{\det(M)} \det(N) \cancel{\det(M)} \quad = \det(N)$$

Invariant building blocks: u has units of velocity

$$F = \det(\partial_j \sigma^I \partial_j \sigma^J) \; ; \quad \det\left(\partial_j \sigma^I \partial_j \sigma^J + \frac{1}{u^2} \partial_t \sigma^I \partial_t \sigma^J\right) = G$$

[and $\det(\partial_t \sigma^I \partial_t \sigma^J)$ ... high order in pert...]

3 Write down the effective action for an ideal fluid.

Invariant BBs: $F = \det(\partial_j \sigma^I \partial_j \sigma^J)$; $G = \det(\underline{\partial_j \sigma^I \partial_j \sigma^J} + \frac{1}{u^2}\partial_t \sigma^I \partial_t \sigma^J)$

$S = \int d^3X\, dt\, \mathcal{L}(F, G)$ is our EFT of a fluid.

Like for solid: $\sigma^I = X_i \delta_i^I + \phi^I$ ← small perturbation

Expand F: $\partial_j \sigma^I \partial_j \sigma^J = \partial_j(X^I + \phi^I)\partial_j(X^J + \phi^J)$

$= (\delta_j^I + \partial_j \phi^I)(\delta_j^J + \partial_j \phi^J)$

$= \delta^{IJ} + \partial^I \phi^J + \partial^J \phi^I + \partial_j \phi^I \partial_j \phi^J$

$$\boxed{\det(1+M) = 1 + \mathrm{tr}(M) + \tfrac{1}{2}[\mathrm{tr}(M^2) - \mathrm{tr}(M)^2]}$$

$F = \det(\delta^{IJ} + \partial^I \phi^J + \partial^J \phi^I + \dots)$

$\approx 1 + 2\partial_I \phi^I + (\partial_I \phi^I)^2 + \dots \approx (1 + \partial_I \phi^I)^2$ ← expansion/compression

$G \approx F + \frac{1}{u^2}\mathrm{tr}(\cancel{\partial_j \sigma^I \partial_j \sigma^J}^{\,\approx \delta^{IJ}} \partial_t \sigma^I \partial_t \sigma^J) = F + \frac{1}{u^2}\partial_t \phi^I \partial_t \phi^I$

↑ $\det(\delta^{IJ} + \frac{1}{u^2}\partial_t \phi^I \partial_t \phi^J)$

Re-cast (up to $(\phi^I)^2$):

$$\boxed{\begin{aligned} f &= \partial_I \phi^I \\ g &= \partial_t \phi^I \partial_t \phi^I \end{aligned}}$$

4 Find the plane wave (sound wave) solutions.

$$\mathcal{L} = A\cdot g - B_1 f - B_2 f^2 + \cdots = A\,\partial_t\phi^I\partial_t\phi^I \cancel{- B_1\partial_I\phi^I} - B_2(\partial_I\phi^I)^2 + \cdots$$

Or: $\mathcal{L} = A\cdot g - B\cdot(F-1) - C\cdot(F-1)^2$

$$= A\cdot\partial_t\phi^I\partial_t\phi^I - B\left(\cancel{2\partial_I\phi^I} + (\partial_I\phi^I)^2\right) - C(2\partial_I\phi^I + \cdots)^2$$

$$= A\,\partial_t\phi^I\partial_t\phi^I - (B+4C)(\partial_I\phi^I)^2.$$

E-L equations:

$$0 = \cancel{\frac{\partial\mathcal{L}}{\partial\phi^I}} - \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\phi^I)} - \partial_j\frac{\partial\mathcal{L}}{\partial(\partial_j\phi^I)} = -2A\partial_t^2\phi_I + (B+4C)2\partial_I\partial_J\phi_J$$

$$\phi_I = a_I e^{ikx-i\omega t}$$

$$A\omega^2 a_I = (B+4C)k_Ik_Ja_J$$

if $a = \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$: $\omega = 0$.

$a = \begin{pmatrix}1\\0\\0\end{pmatrix}$: $\omega^2 = v_s^2k^2$

$\omega = \pm v_s k$

speed of sound

where $v_s = \sqrt{\dfrac{B+4C}{A}}$.