PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 22

Stress tensor of an ideal fluid

October 12

$\boxed{1}$ Review the Lagrangian for an ideal fluid.

Symmetries of an ideal fluid:

$t \to t + \varepsilon_t \qquad X_i \to X_i + \varepsilon_i \qquad X_i \to \overset{O(3)}{R_{ij}} X_j$ ← spatial coords

$\sigma^I \to \xi^I(\sigma)$ obeying $\det\left(\frac{\partial \xi^I}{\partial \sigma^J}\right) = 1.$

$(I = 1,2,3)$ fields ← "volume-preserving coord change"

general function

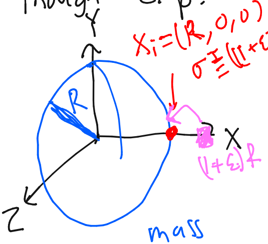
$$\mathcal{L} = Q\Big(\overbrace{\det(\underbrace{\partial_i \sigma^I \partial_i \sigma^J}_{3\times 3 \text{ matrix}})}^{F},\ \det\left(\partial_i \sigma^I \partial_j \sigma^J + u^{-2} \partial_t \sigma^I \partial_t \sigma^J\right)\Big)$$

$$\sigma^I = \underbrace{X_i \delta_i^I}_{\text{equilibrium}} + \phi^I$$ ← small perturbations

$$F \approx (1 + \partial_I \phi^I)^2$$

2 Explain why $\partial_I \phi^I$ relates to mass density.

Thought exp:



Consider: $\sigma^I = (1+\varepsilon) X_i \delta_i^I$ $(\varepsilon \ll 1)$

$\rightarrow \phi^I = \varepsilon X_i \delta_i^I$

Each fluid cell $d^3\sigma$ has mass

$dm = \rho_0 \, d^3\sigma$

How much ~~fluid~~ mass inside ball?

Total mass $M \propto$ # of fluid cells w/ $X_i X_i \le R^2$.

all cells w/ $\sigma^I \sigma^I \le (1+\varepsilon)^2 R^2$

$$= \int_{\sigma^I\sigma^I \le (1+\varepsilon)^2 R^2} d^3\sigma \cdot \rho_0 = \rho_0 \cdot \frac{4\pi}{3}\left[(1+\varepsilon)R\right]^3$$

$$= \underbrace{\left(\frac{4}{3}\pi R^3 \rho_0\right)}_{\rho_0 \cdot V} \cdot (1+\varepsilon)^3$$

Before: $F = (1 + \partial_I \phi^I)^2$

$\partial_I \phi^I = \varepsilon \, \partial_I (X_i \delta_i^I) = \varepsilon \partial_I X^I$

$= 3\varepsilon$

$= \frac{\rho}{\rho_0} - 1$

$\boxed{F \approx \left(\frac{\rho}{\rho_0}\right)^2}$

mass density $\rho = \frac{M}{V} = \rho_0 (1+\varepsilon)^3$

$\approx \rho_0 (1 + 3\varepsilon)$

3 Calculate the (spatial) stress tensor T^{ji}.

For $\underline{\sigma^I = (1+\varepsilon)\chi_i \delta_i^I}$

$$T^{ji} = -\frac{\partial \mathcal{L}}{\partial(\partial_j \sigma^K)} \partial^i \sigma^K + \mathcal{L}\delta^{ji}$$

$(1+\varepsilon)^6$

$\left[\delta^{KL} \cdot (1+\varepsilon)^2\right]^{-1} \cdot 2\delta_j^\ell (1+\varepsilon)$

$$\frac{\partial}{\partial(\partial_j\sigma^K)} \det(\partial_\ell \sigma^I \partial_\ell \sigma^J) = \det(\partial_\ell \sigma^I \partial_\ell \sigma^J) \cdot (\partial_\ell \sigma^K \partial_\ell \sigma^L)^{-1} \cdot 2\partial_j \sigma^L$$

for $\partial_t \sigma = 0$

Recall: $3\varepsilon \approx \frac{\rho}{\rho_0} - 1$; $(1+\varepsilon) \approx \left(\frac{\rho}{\rho_0}\right)^{1/3}$. $\left[\mathcal{L} = Q(\det)\right]$

$$T^{ji} = -\frac{dQ}{d(\det(\cdots))} \frac{\partial(\det\cdots)}{\partial(\partial_j \sigma^K)} \partial_i \sigma^K + Q\delta^{ji},$$ and $\det(\partial_\ell \sigma^I \partial_\ell \sigma^J) \approx \left(\frac{\rho}{\rho_0}\right)^2$;

$$T^{ji} = -Q'\left(\frac{\rho^2}{\rho_0^2}\right)\left(\frac{\rho}{\rho_0}\right)^{2-2/3} \underbrace{2\partial^i \sigma^K \partial^j \sigma^K}_{2(1+\varepsilon)^2 \delta^{ji} = 2\left(\frac{\rho}{\rho_0}\right)^{2/3}\delta^{ji}} + Q\left(\frac{\rho^2}{\rho_0^2}\right)\delta^{ji}$$

pressure

$$\hookrightarrow T^{ji} = \delta^{ji}\left[-Q'(\rho^2/\rho_0^2) \cdot 2\frac{\rho^2}{\rho_0^2} + Q\left(\frac{\rho^2}{\rho_0^2}\right)\right] = \delta^{ji} \cdot P(\rho)$$

4 Calculate the energy density $\underline{T^{tt}}$. Discuss the relationship to thermodynamics.

T^{tt} = energy density. Calculate neglecting $\partial_t \sigma^I$.
(thermodynamics only)

$$T^{tt} = \mathcal{L}\eta^{tt} = -\mathcal{L} = -Q\left(\frac{\rho^2}{\rho_0^2}\right).$$

Hence $Q = -$ energy density

Thermodynamics:

$$\left.\frac{dE}{dV}\right|_{\text{fixed } \cancel{S}\, M} = -P$$

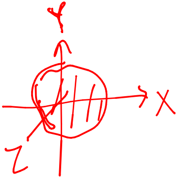
E = total energy

V = volume

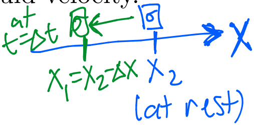
$$E = V\cdot\left(-Q\left(\frac{\rho^2}{\rho_0^2}\right)\right)$$
$$= -VQ\left(\left(\frac{M}{\rho_0 V}\right)^2\right)$$

$$\frac{dE}{dV} = -Q\left(\left(\frac{M}{\rho_0 V}\right)^2\right) - VQ'\left(\left(\frac{M}{\rho_0 V}\right)^2\right)\frac{-2M^2}{\rho_0^2 V^3}$$
$$= -Q\left(\frac{\rho^2}{\rho_0^2}\right) + 2Q'\left(\left(\frac{\rho}{\rho_0}\right)^2\right)\left(\frac{\rho}{\rho_0}\right)^2$$
$$= -P(\rho)$$

agrees w/ T^{ji}



5 Give the fluid's equations of motion in terms of a fluid velocity.



Define fluid velocity: $v^I = -\partial_t \phi^I$

why minus?

$\partial_t \rho = \partial_t(\rho - \rho_0) = \rho_0 \partial_t \partial_I \phi^I = -\partial_I(\rho_0 v^I)$

Hence $\partial_t \rho + \partial_I(\rho_0 v^I) = 0$

For small $\Delta\rho$, v^I, continuity equation!

$\sigma(X_1, \Delta t) = X_2$

$= X_1 + \Delta X$

$= \sigma_1(X_1, 0) + \frac{\Delta X}{\Delta t}\Delta t$

$\Rightarrow \partial_t \sigma(X_1) > 0$ while fluid has vel < 0.

Use momentum conservation:

$$T^{ti} = -\frac{\partial \mathcal{L}}{\partial(\partial_t \sigma^K)} \partial_i \sigma^K \propto -\partial_t \phi^i \propto v^i$$

momentum density

for ordinary fluids: $T^{ti} \approx \rho v^i$

Momentum Noether equation: $[X_i \to X_i + \varepsilon_i]$

$$0 = \partial_t T^{ti} + \partial_j T^{ji} = \partial_t(\rho v^i) + \partial_i P(\rho)$$

To linear order in small v^i, $\rho - \rho_0$, these are Euler equations