PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 23

Hamilton's equations

October 14

$\boxed{1}$ How can we convert Lagrange's equations into first order equations?

Lagrangian mechanics for 1d system: $S = \int dt\, L(x,\dot{x})$

E-L equations: $0 = \frac{\delta S}{\delta x(t)} = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$

Take 2nd order ODE for $x(t)$ $\rightarrow$ 2 1st order ODEs for $x(t), p(t)$

$p = \frac{\partial L}{\partial \dot{x}}\Big|_x$ = "canonical momentum"

$\dot{p} = \frac{\partial L(x,\dot{x})}{\partial x}\Big|_{\dot{x}}$ [bad? $\frac{\partial L}{\partial x}$ is evaluated at fixed $\dot{x}$...]

the Hamiltonian

H

$$\dot{p} = \frac{\partial}{\partial x} L(x, \dot{x}(x,p))\Big|_p - \frac{\partial L}{\partial \dot{x}}\Big|_x \frac{\partial \dot{x}}{\partial x}\Big|_p = \frac{\partial L}{\partial x}\Big|_p - p\frac{\partial \dot{x}}{\partial x}\Big|_p = -\frac{\partial}{\partial x}\underbrace{(p\dot{x} - L)}_{H}\Big|_p$$

$\frac{\partial L}{\partial x}\Big|_x + \frac{\partial L}{\partial \dot{x}}\Big|_x \frac{\partial \dot{x}}{\partial x}\Big|_p$

$\dot{x} = ?$

$$\frac{\partial H}{\partial p}\Big|_x = \frac{\partial}{\partial p}(p\dot{x} - L)\Big|_x = \dot{x} + \cancel{p\frac{\partial \dot{x}}{\partial p}\Big|_x} - \cancel{\underbrace{\frac{\partial L}{\partial \dot{x}}\Big|_x}_{=p} \frac{\partial \dot{x}}{\partial p}\Big|_x}$$

$$\boxed{\dot{x} = \frac{\partial H}{\partial p}} \quad \boxed{\dot{p} = -\frac{\partial H}{\partial x}}$$ Hamilton's equations.

2 Summarize Hamilton's equations.

Hamilton's equations: $\boxed{\dfrac{\partial H}{\partial p} = \dot{x} \quad \text{and} \quad -\dfrac{\partial H}{\partial x} = \dot{p}}$

$$H(p,x) = p\dot{x} - L \,, \quad \boxed{p = \left.\frac{\partial L}{\partial \dot{x}}\right|_x}$$

This prescription OK if $p(\dot{x})\big|_x$ is invertible.

Assume $\left.\dfrac{\partial p}{\partial \dot{x}}\right|_x > 0$ or $\underbrace{\left.\dfrac{\partial^2 L}{\partial \dot{x}^2}\right|_x}_{= \, m>0} > 0$. (eg. $L = \frac{1}{2}m\dot{x}^2 - V(x)$)

Generalizes nicely to $L(x_i, \dot{x}_i)$ $i = 1, \dots, n$:

$$H = p_i \dot{x}_i - L \qquad -\frac{\partial H}{\partial x_i} = \dot{p}_i \quad \text{and} \quad \frac{\partial H}{\partial p_i} = \dot{x}_i$$

$$p_i = \left.\frac{\partial L}{\partial \dot{x}_i}\right|_x$$

3 Describe a variational principle for Hamiltonian mechanics.

Write $$S = \int dt \; L(x_i, \dot{x}_i) = \int dt \left[p_i \dot{x}_i - H(x,p) \right]$$

$$\frac{\delta S}{\delta p_i(t)} = \frac{\partial}{\partial p_i}\left[p_i \dot{x}_i - H \right] - \cancel{\frac{\partial}{\partial \dot{p}_i}\left[\; \right]} = 0$$

$$= \underline{\dot{x}_i - \frac{\partial H}{\partial p_i} = 0}$$

$$\frac{\delta S}{\delta x_i(t)} = -\frac{\partial H}{\partial x_i} - \frac{d}{dt}\frac{\partial (p_j x_j)}{\partial \dot{x}_i} = -\frac{\partial H}{\partial x_i} - \dot{p}_i = 0 .$$

4 Study the harmonic oscillator with Hamiltonian mechanics.

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$p = m\dot{x} = \frac{\partial L}{\partial \dot{x}} \quad ; \quad \dot{x} = \frac{p}{m}$$

Noether constant for $t \to t + \varepsilon$.

$= \text{energy!}$

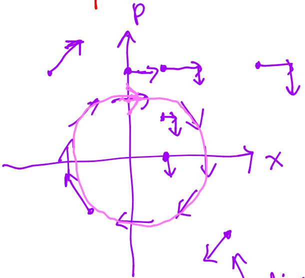
$$H = p\dot{x} - L = p \cdot \frac{p}{m} - \left[\frac{1}{2} m \left(\frac{p}{m}\right)^2 - \frac{1}{2} k x^2\right] = \overbrace{\frac{p^2}{2m} + \frac{1}{2} k x^2}.$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial x} = -kx$$

$$\left[\ddot{x} = -\frac{k}{m} x \right.$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\left. x(t) = A\cos(\omega t) + B\sin(\omega t) \right]$$



direction of $(\dot{x}, \dot{p})$

5 Study a charged particle in a magnetic field.

$$L = \frac{1}{2} m \dot{x}_i \dot{x}_i + q A_i(x) \dot{x}_i$$ (non-relativistic)

$$\vec{B} = \nabla \times \vec{A}. \qquad B_i = \varepsilon_{ijk} \partial_j A_k$$

$$\dot{x}_i = \frac{p_i - qA_i}{m} \checkmark$$

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i(x).$$ NOT "kinetic" momentum... canonical

Legendre transform: $$H = p_i \dot{x}_i - L = p_i\left(\frac{p_i - qA_i}{m}\right) - \frac{m}{2}\frac{(p_i - qA_i)(p_i - qA_i)}{m^2} - qA_i\dot{x}_i$$

$$H = \frac{1}{2m}(p_i - qA_i)(p_i - qA_i)$$

$$\frac{\partial H}{\partial p_i} = \dot{x}_i = \frac{p_i - qA_i}{m}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = -\frac{1}{m}(p_j - qA_j)\left(-q\frac{\partial A_j}{\partial x_i}\right)$$

$$\dot{p}_i = q\frac{\partial A_j}{\partial x_i}\dot{x}_j$$

$$= \frac{d}{dt}(m\dot{x}_i + qA_i) = m\ddot{x}_i + q\frac{\partial A_i}{\partial x_j}\dot{x}_j$$

$$m\ddot{x}_i = \underbrace{q\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j}_{q\varepsilon_{ijk}\dot{x}_j B_k}$$

$$m\ddot{\vec{x}} = q\dot{\vec{x}} \times \vec{B}$$