PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 24

Poisson brackets

October 17

1 Define the symplectic form and it's properties.

coords (canonical): (q^i, p^i) $\quad i=1,\dots,N$

first order: $\underline{\dot{q}^i = \frac{\partial H}{\partial p^i}}$ and $\underline{\dot{p}^i = -\frac{\partial H}{\partial q^i}}$

Define: $\xi^I = \begin{pmatrix} q^1 \\ \vdots \\ q^N \\ p^1 \\ \vdots \\ p^N \end{pmatrix}$.

2N-comp.

Then $\dot{\xi}^I = \omega^{IJ}\frac{\partial H}{\partial \xi^J} = \omega^{IJ}\partial_J H$

where $\omega^{IJ} = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & \underbrace{0}_{N\times N} \end{array}\right)$

Symplectic form (2-form)

Define (like w/ $\eta^{\mu\nu}$): ω_{IJ} such that $\omega_{IJ}\omega^{JK} = \delta^K_I$, $\omega^{IJ}\omega_{JK} = \delta^I_K$

$$\omega_{IJ} = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right)$$

2 Define the Poisson bracket, and discuss its properties.

$\omega^{IJ} = -\omega^{JI}$

Define Poisson bracket: $[f,g] = \partial_I f\, \omega^{IJ} \partial_J g$

$\delta^K_I = \partial\xi^K / \partial\xi^I$

in canonical: $[f,g] = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q^i}$.

$[\xi^K, \xi^L] = \partial_I \xi^K \omega^{IJ} \partial_J \xi^L = \delta^K_I \omega^{IJ} \delta^L_J = \omega^{KL}$.

$\hookrightarrow$ canonical: $[q^i, q^j] = 0$; $[p^i, p^j] = 0$; $[q^i, p^j] = \delta^{ij}$.

Connect to QM: $[x,p] = i\hbar$

More generally: $[f,g]_{QM} = i\hbar [f,g]_{PB}$ (up to operator ordering in QM)

Important properties:

1) $[f,g] = -[g,f]$

2) $[f, g_1 g_2] = g_1 [f, g_2] + [f, g_1] g_2$

3) (Jacobi identity): $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$.

Lie algebra.

3 What is the "Poisson bracket equation of motion"?

For any function f: $\boxed{\dot{f} = [f, H] + \frac{\partial f}{\partial t}}$ ✓

Proof: 1) Holds for $f = \xi^K$: $\dot{\xi}^K = [\xi^K, H] = \partial_I \xi^K \omega^{IJ} \partial_J H$

$= \omega^{KJ} \partial_J H$ ✓

Hence: $\dot{q}^k = \frac{\partial H}{\partial p^k}$, and $\dot{p}^k = -\frac{\partial H}{\partial q^k}$

2) $$\frac{d}{dt} f(q,p,t) = \frac{\partial f}{\partial t} + \underbrace{\frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial p^i}\dot{p}^i}_{\partial_I f \dot{\xi}^I = \partial_I f\, \omega^{IJ} \partial_J H = [f, H]}$$

If $\frac{\partial f}{\partial t} = 0$, f is a conserved quantity if/only if $[f, H] = 0$.

leads to $\rightarrow$ converse of Noether's Thm.

4 Discuss when a problem has rotational symmetry.

Recall: Lag. formulation, we have rotational sym if

$L(x_i x_i, \dot{x}_i \dot{x}_i, \dot{x}_i x_i, \ldots)$

(once working w/ q, p ... raise/lower freely)

In Ham picture: $[L_i, H] = 0$

$\hookrightarrow H(x_i p_i, x_i x_i, p_i p_i)$

Angular momentum: $L_i = \varepsilon_{ijk} x_j p_k$

$\hookrightarrow L_x = y p_z - z p_y$

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$.

ε_{ijk} is Levi-Civita:

$\varepsilon_{xyz} = \varepsilon_{yzx} = \varepsilon_{zxy} = 1$

$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}$

$\hookrightarrow \varepsilon_{yxz} = \varepsilon_{zyx} = \varepsilon_{xzy} = -1$

$\varepsilon_{xxi} = 0 = -\varepsilon_{xxi}$

5 Evaluate Poisson brackets involving angular momenta

$[L_i, p_j] = \varepsilon_{ijk} p_k$; ✓

$$[\varepsilon_{ilm} x_l p_m, p_j]$$
$$= \varepsilon_{ilm} [x_l p_m, p_j]$$
$$= \varepsilon_{ilm} \left(x_l [p_m, p_j] + [x_l, p_j] p_m \right)$$
$$= \varepsilon_{ilm} \left(\cancel{x_l \cdot 0} + \delta_{lj} p_m \right)$$
$$= \varepsilon_{ijk} p_k$$

On HW9: $[L_i, x_j] = \varepsilon_{ijk} x_k$

$[L_i, L_j] = \varepsilon_{ijk} L_k$,

Hence: if we want $[L_i, H] = 0$, then $H(x_i x_i, p_i p_i, \ldots)$

invariant!

Try: $[L_i, p_j p_j] =$
$$2 p_j [L_i, p_j] =$$
$$2 \varepsilon_{ijk} p_j p_k = 0.$$

Also: $x_i x_i$, $x_i p_i$ invariants.

Lemma: If $[a, b] = 0$, then $[a, f(b)] = 0$.

Suppose $f(b) = f_0 + f_1 b + f_2 b^2 + \cdots$

$$[a, f(b)] = \cancel{[a, f_0]} + f_1 [a, b] + 2 f_2 b \cdot [a, b] + \cdots$$
$$= [a, b] \cdot f'(b) = 0.$$