PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 25

Symplectic manifolds and canonical transformations

October 19

1 Define a symplectic manifold (heuristically).

Lagrangian:

1) find config. space: M, n-dim manifold if n DOFs. (e.g. $\mathbb{R}^n$, $SO(3)$...)

2) $S = \int dt\, L$, extremize S.

Hamiltonian:

(M, ω) is symplectic manifold

1) find phase space: manifold M
M is $2n$-dim if n DOFs
local coords: (q_i, p_i)

2) Find symplectic form: $\omega_{IJ} = -\omega_{JI}$ $\left[= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]$ ($n \times n$)

- closed: $\partial_I \omega_{JK} + \partial_J \omega_{KI} + \partial_K \omega_{IJ} = 0$.
- non-degenerate: ω^{-1} exists. (ω^{IJ})

3) Pick "any" function $H : M \to \mathbb{R}$ (Hamiltonian).

$\to \dot{f} = [f, H] + \cancel{\frac{\partial f}{\partial t}}$

where $[f, H] = \omega^{IJ} \partial_I f\, \partial_J H$.

$\boxed{2}$ Define a canonical transformation.

Goal: find coord. transforms. that leave $\underbrace{(M, \omega)}^{\text{symplectic manifold.}}$ invariant.

Such a coord. transform is called canonical (CT)

How does ω transform? To figure out, $\frac{\partial f}{\partial t} = 0$, and

$$\dot{f} = [f, H] = \omega^{IJ} \partial_I f \partial_J H = \omega^{IJ}(\xi) \frac{\partial f}{\partial \xi^I} \frac{\partial H}{\partial \xi^J}$$

If transform $\xi^I \to \xi^I(\eta)$ [assume invertible]

Need: $\dot{f} = \omega^{IJ}(\xi(\eta)) \frac{\partial f}{\partial \eta^I} \frac{\partial H}{\partial \eta^J}$ use chain rule:

$$= \omega^{IJ} \left(\frac{\partial \xi^K}{\partial \eta^I} \frac{\partial f}{\partial \xi^K} \right) \left(\frac{\partial \xi^L}{\partial \eta^J} \frac{\partial H}{\partial \xi^L} \right) = \underbrace{\left(\frac{\partial \xi^K}{\partial \eta^I} \omega^{IJ} \frac{\partial \xi^L}{\partial \eta^J} \right)} \frac{\partial f}{\partial \xi^K} \frac{\partial H}{\partial \xi^L}$$

Invert:

$$\omega_{KL} = \frac{\partial \eta^I}{\partial \xi^K} \omega_{IJ} \frac{\partial \eta^J}{\partial \xi^L}$$

Need

$$\boxed{\omega^{KL} = \frac{\partial \xi^K}{\partial \eta^I} \omega^{IJ} \frac{\partial \xi^L}{\partial \eta^J}}$$

3 Give an example of a canonical transformation for the harmonic oscillator.

harmonic oscillator: $H = \frac{p^2}{2} + \frac{q^2}{2}$ $(m=\omega=1)$ (Step 3)

Steps 1 & 2: $M = \mathbb{R}^2$ & $\omega_{IJ} = \begin{array}{cc} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{array}{c} x \\ p \end{array} \end{array}$

Ex. 1: $p' = p + 1$ or $x' = x + 1$.

These are CTs but H changes.

Ex. 2: $p' = \lambda p$ and $x' = \frac{1}{\lambda} x$

$\omega_{px} = \omega_{p'x'} = \frac{\partial p'}{\partial p} \omega_{p'x'} \frac{\partial x'}{\partial x} = \lambda \cdot 1 \cdot \frac{1}{\lambda} = 1$. ✓

Ex. 3: $\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$.

- Since this is not: $H = \frac{(p')^2 + (q')^2}{2}$.
- can check it's CT explicitly as above...

$\omega_{px} = \partial(\xi')^I / \partial p \; \omega_{IJ} \; \partial(\xi')^J / \partial x$

4 Show that under time evolution, the mapping $\xi^I \to \xi^I(t)$ is a canonical transformation. e.g. Ex. 3.

$$\boxed{\xi^I(0) \xrightarrow[\text{CT}]{} \xi^I(dt) \quad \text{Goal:}} \xrightarrow[\text{CT}]{} \xi^I(2dt) \xrightarrow[\text{CT}]{} \cdots \xrightarrow[\text{CT}]{} \xi^I(t)$$

Use Hamilton's eq: $\dot{\xi}^I = [\xi^I, H]$

$$\xi^I(dt) \approx \xi^I(0) + dt\,[\overbrace{\xi^I, H}^{\omega^{IJ}\partial_J H}]$$

$$\frac{\partial \xi^I(dt)}{\partial \xi^J(0)} = \delta^I_J + dt \cdot \omega^{IK} \partial_J \partial_K H$$

Check CT: $\omega_{KL} = \frac{\partial \xi^I(dt)}{\partial \xi^K(0)} \omega_{IJ} \frac{\partial \xi^J(dt)}{\partial \xi^L(0)}$

$$\cancel{\omega_{KL}} = \cancel{\delta^I_K \omega_{IJ} \delta^J_L} + dt\Big[\delta^I_K \underbrace{\omega_{IJ}\omega^{JM}}_{=\delta^M_I} \partial_M \partial_L H + \omega^{IM} \partial_M \partial_K H \, \omega_{IJ} \delta^J_L\Big]$$

$$= -\delta^M_J$$

$$+ dt\,[\partial_K \partial_L H - \partial_K \partial_L H] = 0.$$