PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 26

Generating functions of canonical transformations

October 21

$\boxed{1}$ Review the definition of a canonical transformation.

Hamiltonian mech defined on symplectic manifold

(M, ω)

(phase space) $2n$-dimens. $\uparrow$ $\uparrow$ $\omega_{IJ} = -\omega_{JI}$

invertible

Canonical transformation = coord changes that leave ω unchanged.

Consider: $M = \mathbb{R}^2$, $\omega = \begin{array}{cc} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \begin{array}{c} x \\ p \end{array} \end{array}$

$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} x+1 \\ p \end{pmatrix}$. $\quad \begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \quad \begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$

canonical $\quad$ canonical

$\underset{(2n)}{\xi^I} \to \xi^I(\eta)$ is canonical if it's invertible, and

$$\omega_{IJ}(\xi(\eta)) = \frac{\partial \xi^K}{\partial \eta^I} \omega_{KL}(\xi(\eta)) \frac{\partial \xi^L}{\partial \eta^J} \to \dot{f} = [f, H]$$

true in both coords. ξ & η.
[if transform. is t-independent]

2 Show that canonical transformations must come from a single generating function (up to topological obstructions...).

$$\omega_{IJ}(\xi) = \frac{\partial \alpha_J}{\partial \xi^I} - \frac{\partial \alpha_I}{\partial \xi^J} \longrightarrow \text{e.g. if } \omega = \begin{array}{c} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array}\begin{array}{c} x \\ p \end{array}$$

math fact from topology.

$$\alpha = \begin{pmatrix} p \\ 0 \end{pmatrix}; \quad \alpha = p\,dx$$

Suppose we have CT $(q_i, p_i) \to (Q_i, P_i)$, and

$\alpha = p_i\, dq_i$ OR $A = P_i\, dQ_i$

If: $\omega(p,q) - \omega(P,Q) = 0 = \Delta\omega$. $\Delta\omega_{IJ} = \frac{\partial \Delta\alpha_J}{\partial \xi^I} - \frac{\partial \Delta\alpha_I}{\partial \xi^J} = 0$.

Hence $\Delta\alpha_J = \partial_J F$. Write $\Delta\alpha = dF = p_i\, dq_i - P_i\, dQ_i$.

$(\not\partial\alpha)$

3 Discuss a type 1 canonical transformation.

Suppose $F(q_i, Q_i)$, $Q_i(q,p)$, H are all t-independent.

$dF = p_i\,dq_i - P_i\,dQ_i$: $\boxed{p_i = \frac{\partial F}{\partial q_i} \text{ and } P_i = -\frac{\partial F}{\partial Q_i}}$

Assume "invertible": one (q, Q) for each p, or P.

Type 1: (q, Q) are independent. $H_{old}(Q_i, P_i) = H(q(Q,P), p(Q,P))$

Example: $M = \mathbb{R}^2$, $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $Q = \cos\theta\, q + \sin\theta\, p$, $P = \cos\theta\, p - \sin\theta\, q$

$0 < \theta < \pi$

$p = \frac{Q - q\cos\theta}{\sin\theta} = \frac{\partial F}{\partial q}$ $= \csc\theta \cdot Q - q\cot\theta$.

$F = -\frac{q^2}{2}\cot\theta + qQ\csc\theta - \frac{Q^2}{2}\cot\theta$.

Similarly:

$P = \cot\theta\, Q - \csc\theta \cdot q$

If CT F has t-dep:

$H_{new}(Q,P) = H_{old}(Q,P) + \frac{\partial F}{\partial t}$

4 What are type 2,3,4 canonical transformations?

Type 2 CT: (q, P) independent. Legendre transform

$$F^2(q,P,t) = F(q, Q(q,P,t), t) + P_i Q_i$$

$$H_{new} = H_{old} + \frac{\partial F^2}{\partial t}$$

$$p_i = \frac{\partial F^2}{\partial q_i}, \quad Q_i = \frac{\partial F^2}{\partial P_i}$$

Type 3 CT: (p, Q)

$$F^3 = F - p_i q_i \qquad [dF^3 = dF - p_i dq_i - q_i dp_i]$$

etc... $q_i = -\frac{\partial F^3}{\partial p_i}$, $P_i = -\frac{\partial F^3}{\partial Q_i}$ etc...

Type 4 CT: (p, P) $\quad F^4 = F - p_i q_i + P_i Q_i$

5 Discuss the one-parameter family of canonical transforms generated by a function f.