PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 27

Noether's Theorem on symplectic manifolds

October 24

1 Prove Noether's Theorem in Hamiltonian mechanics.

continuous symmetry $\longrightarrow$ conservation law

Assume $\frac{\partial f}{\partial t} = 0$.

?? in Hamiltonian?

$\dot{f} = 0 = [f, H]$

continuous family of CTs leave H invariant.

cf lec. 25 not $\neq H$

$$\frac{d\xi^I(s)}{ds} = [\xi^I(s), g]$$

$\xi^I(0) \rightarrow \xi^I(s)$ is C.T.

Suppose $H(\xi^I(s)) = H(\xi^I)$

$$\frac{dH}{ds} = 0$$

Evolution in s generated by g. So

$$\frac{dH}{ds} = \frac{\partial H}{\partial \xi^I} \frac{d\xi^I}{ds} = [H, g] = 0$$

Converse: if $[H, g] = 0$, then there's CTs $\xi^I \rightarrow \xi^I(s)$

w/ $\frac{d\xi^I(s)}{ds} = [\xi^I(s), g] \Longrightarrow$ continuous symmetry

2 Show that conserved quantities generate a Lie algebra.

Claim: if f, g are conserved, so is $[f,g]$.

if $[f,H] = 0 = [H,f]$

Proof: (Jacobi identity): $\frac{d}{dt}[f,g] = \big[[f,g],H\big]$

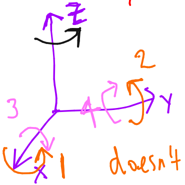
$$= -\big[\cancel{[g,H]},f\big] - \big[\cancel{[H,f]},g\big] = 0$$

continuous symmetries form Lie algebra: $f, g \in$ alg means $[f,g] \in$ alg.

Ex: angular momentum (L_i): $[H, L_i] = 0$

$[L_i, L_j] = \varepsilon_{ijk} L_k$

If L_x, L_y are conserved, so is L_z:



doesn't change H.

Algebraic structure ensures continuous symmetry group.

$$SI \xrightarrow{\text{sym}} SI' \xrightarrow{\text{sym}} SI''$$

sym

3 What are the implications of momentum conservation?

2 particles in 1d:

$p_{tot} = p_1 + p_2$ is conserved.

→ deduce continuous symmetry.

$$\underbrace{\frac{d\xi^I}{ds} = [\xi^I, p_{tot}]}$$

$$\xi^I = \begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix} \qquad [\xi^I, p_{tot}] = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

since $[x_i, p_j] = \delta_{ij}$.

Deduce a continuous symmetry:

$$\begin{pmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 + s \\ x_2 + s \\ p_1 \\ p_2 \end{pmatrix}$$

global translation

Spirit of effective theory:
what is most general H?

Look for invariant building blocks:
$H(p_1, p_2, x_1 - x_2)$.

Example: diatomic mol.

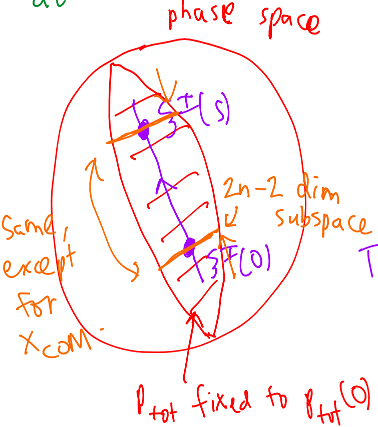
$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \underbrace{V(x_1 - x_2)}_{\frac{1}{2}k(x_2 - x_1 - a)^2}$$

4 Explain how a conserved quantity removes two degrees of freedom from phase space.

If you have conserved p_{tot} ... remove pair of phase space coords.

Ignorable DOF = $p_1 + p_2$ & COM: $x_1 + x_2$ $(m_1 = m_2)$

$$\frac{dp_{tot}}{dt} \overset{\text{assume!}}{=} 0 \qquad \text{AND: } \underbrace{\frac{dx_{com}}{dt} = \frac{p_{tot}}{2m} = \text{const.}}$$



Know H can't depend on x_{com}:

$[x_{com}, p_{tot}] \neq 0$. $\qquad (x_{com} \to x_{com} + s)$

Use $\underbrace{\frac{d\xi^I}{ds} = [\xi^I, p_{tot}]}$

Take points $\{\xi^I(s) \text{ for any } s\}$, identify them together

mathematics: 2n-2-dim subspace will be symplectic