PHYS 5210
Graduate Classical Mechanics
Fall 2022

Lecture 28
The Hamilton-Jacobi equation

October 26

$\boxed{1}$ Review the type 1 canonical transformation.

canonical transformation: coord change $(\utilde{q_i}, p_i) \to (\underline{Q_i}, P_i)$

preserves ω [preserves Poisson Bs]

cf lec 26: Type 1 CT: (q_i, Q_i) are independent

What doesn't work: $Q_i = q_i$; $P_i = p_i + \frac{\partial \chi(q)}{\partial q_i}$ (cf HW 10)

Example: $Q_i = q_i + p_i$; $P_i = p_i$. $[Q_i, P_j] = [q_i, p_j] = \delta_{ij}$.

$\underline{p_i = Q_i - q_i}$

"Generating functions" $F(q_i, Q_i, t)$ such that:

$$p_i = \frac{\partial F}{\partial q_i} \; ; \quad P_i = -\frac{\partial F}{\partial Q_i} \; ; \quad \underline{H_{new}(Q_i, P_i, t) = H(q,p,t) + \frac{\partial F}{\partial t}}$$

2 Derive the Hamilton-Jacobi equation.

Goal: Find Type 1 CT such that $H_{new} = 0$.
Call generating function of this special CT: S

$$\text{If } H_{new} = 0 = \underbrace{H(q_i, \overset{(q,Q)}{p_i}, t) + \frac{\partial \underline{S(q_i, Q_i, t)}}{\partial t}}$$

Claim: Q_i's and P_i's are all constants of integration.

$$\dot{Q}_i = \frac{\partial H_{new}}{\partial P_i} = [Q_i, \cancelto{0}{H_{new}}] = 0. \checkmark \quad , \quad \text{similarly: } \dot{P}_i = 0.$$

$$\boxed{0 = \frac{\partial \underline{S(q_i, Q_i, t)}}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right)}$$ Hamilton-Jacobi equation

Old: Hamilton: 2n first order ODEs.
HJ: PDE in n space + 1 time dimension.

Why???

3 How is it solved with separation of variables?

Most problems NOT solvable this way...

If $\frac{\partial H_{old}}{\partial t} = 0$ ... $0 = \frac{\partial S}{\partial t} + H(q_i, \frac{\partial S}{\partial q_i})$

Ansatz: $S = -Et + W(q_i)$: $0 = -E + H(q_i, \frac{\partial S}{\partial q_i})$, or $E = H$

Separation of variables: difference is here we won't pick

$$\cancel{S = S_1(q_1)\, S_2(q_2) \cdots}$$

$$S = -Et + W_1(q_1) + W_2(q_2) + \cdots$$

4 Set up the Kepler problem in the Hamilton-Jacobi formulation.

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}.$$

Canonical coords: $(r, \theta, p_r, p_\theta)$

H-J

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 - \frac{k}{r} = 0.$$

Choose $S = -Et + W(r, \theta)$:

$$E = \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \theta}\right)^2 - \frac{k}{r}$$

Next: $W(r, \theta) = W_r(r) + W_\theta(\theta)$:

$$E = \text{const.} = \frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \underbrace{\frac{1}{2mr^2}\overset{p_\theta^2}{\cancel{\left(\frac{dW_\theta}{d\theta}\right)^2}}}_{\text{only depend on } r:} - \frac{k}{r}$$

Q_1, Q_2 are both consts.

$\frac{dW_\theta}{d\theta} = p_\theta$, $W_\theta = p_\theta \cdot \theta$. So far: $S = -\overset{Q_1}{\boxed{E}}t + \overset{Q_2}{\boxed{p_\theta}}\theta + W_r(r)$

5 Solve this Kepler problem.

$$E = \frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2 + \frac{p_\theta^2}{2mr^2} - \frac{k}{r}$$

$$\hookrightarrow \frac{dW_r}{dr} = \sqrt{2mE + \frac{2mk}{r} - p_\theta^2/r^2} \rightarrow W_r(r) = \int dr \sqrt{2mE + \frac{2km}{r} - p_\theta^2/r^2}$$

Don't care about S . . . we do know P_1 & P_2 are const.

$$\cancel{P_1 = -\frac{\partial S}{\partial Q_1}} \; ; \quad \boxed{P_2 = -\frac{\partial S}{\partial Q_2}} \qquad Q_2 \rightarrow p_\theta$$

$$\text{const.} = -\frac{\partial S}{\partial p_\theta} = -\theta - \frac{\partial W_r}{\partial p_\theta}$$

$$= -\theta - \int dr \frac{1}{2\sqrt{2mE + \frac{2km}{r} - p_\theta^2/r^2}} \cdot \left(-\frac{2p_\theta}{r^2}\right)$$

$\hookrightarrow \theta' - \theta_0 = \int_{r_0}^{r'} dr \, . . .$ gives $\theta'(r') \rightarrow$ Shape of orbit