PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 29

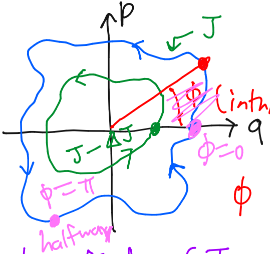
Action-angle variables

October 28

1 Describe the structure of the action-angle variables.

Generic H for 1 particle (2 phase space DOF): $H(q,p)$

Assume $\frac{\partial H}{\partial t} = 0$.



first coord. J label curve along which system moves

(intuition only)

ϕ and $\phi + 2\pi$ identified...

Goal: find CT from $(q,p) \longrightarrow (\phi, J)$.

angle, action

$$\frac{dJ}{dt} = 0 = [J,H] = -\frac{\partial H}{\partial \phi}$$

so $H = H(J)$

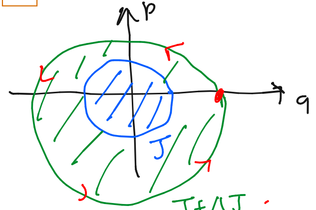
$$\frac{d\phi}{dt} = \dot{\phi} = \frac{\partial H}{\partial J}$$ b/c $\frac{\partial H}{\partial J}$ ind. of t.

$$\ddot{\phi} = 0$$

$$\phi(t) = \phi(0) + \omega_0 t$$

where $\omega_0 = \frac{\partial H}{\partial J}$

2 Define the action variable J.



Choose: $J = \frac{1}{2\pi}\oint p\,dq$

$= \frac{1}{2\pi}$[Area enclosed by orbit].

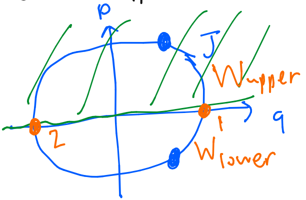
$J+\Delta J$: Area$(J+\Delta J) >$ Area(J)

$J+\Delta J > J$

From geometric construction: $\frac{dJ}{dt}=0$ (again!)

3 Describe the canonical transformation to action-angle variables.

Goal: Type 2 CT from $(\underline{q}, p) \to (\phi, \underline{J})$.



NOT $\underline{\text{globally}}$ type 2 . . .

Locally . . . (q, J) independent . . .

For type 2 CT: generating function $W(q, J)$

$$p = \frac{\partial W}{\partial q} \quad \text{and} \quad \phi = \frac{\partial W}{\partial J}.$$

When separating variables w/ HJ equation: $S = W - Et$

$$E = H\left(q, \frac{\partial W}{\partial q}\right) \therefore \text{ solve for } \frac{\partial W}{\partial q} \; : \quad 2\pi J = \underbrace{\oint \frac{\partial W}{\partial q}\, dq}$$

naively:

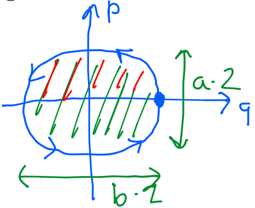
$$= W(q_f) - W(q_f) = 0$$

$$2\pi J = \int_1^2 \frac{\partial W_{up}}{\partial q}\, dq + \int_2^1 \frac{\partial W_{low}}{\partial q}\, dq = f(E) \leftarrow H^{-1}$$

$$E = H(J) \quad \text{or} \quad J = H^{-1}(E)$$

4 Solve the harmonic oscillator using action-angle variables.

$$H = \frac{p^2}{2m} + \frac{1}{2} m w^2 q^2 \quad \leftarrow \text{ellipse}$$



$$J = \frac{1}{2\pi} \oint dq\, p = \frac{\text{Area}}{2\pi} = \frac{\pi a b}{2\pi} = \frac{ab}{2}$$

Goal: $H(J) = E$; $J = H^{-1}(E)$

$$\left.\begin{aligned} E &= \frac{1}{2} m\omega^2 b^2 \\ &= \frac{1}{2m} a^2 \end{aligned}\right\} \quad \begin{aligned} b &= \sqrt{2E/m\omega^2}, \\ a &= \sqrt{2mE} \end{aligned} \quad \text{so} \quad \begin{aligned} J &= \frac{1}{2}\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}} \\ &= \frac{E}{\omega} \end{aligned}$$

Hence: $E = \boxed{H(J) = \omega J}$

$$\dot{J} = 0 = -\frac{\partial H}{\partial \phi}$$

and $\dot{\phi} = \dfrac{\partial H}{\partial J} = \text{const.} = \omega$, so $\underline{\phi(t) - \phi(0) = \omega t.}$