PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 3

Relativistic motion

August 26

1 Review the postulates of special relativity.

Spacetime coords:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = x^\mu$$

Postulate #1: Symmetries are $\underbrace{\text{const}}$ / linear transform.

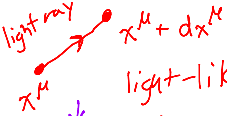
Translations: $x^\mu \to x^\mu + a^\mu$ (const.)

repeated ν: $\sum_\nu$

Linear! (Rotation & Boost) $x^\mu \to \Lambda^\mu{}_\nu x^\nu$

$$\delta^\mu{}_\nu = \eta^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Postulate #2: (inertial) observers see speed of light c.



light-like traject.:

$$0 = -(c\,dt)^2 + dx^2 + dy^2 + dz^2$$

$$\hookrightarrow 0 = dx_\mu dx^\mu$$

$$dx_\mu = \begin{pmatrix} -c\,dt \\ dx \\ dy \\ dz \end{pmatrix} \qquad dx^\mu = \begin{pmatrix} c\,dt \\ dx \\ dy \\ dz \end{pmatrix}$$

$$\hookrightarrow = \eta_{\mu\nu} dx^\nu \qquad \hookrightarrow = dx_\nu \eta^{\mu\nu}$$

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2 What are the Lorentz transformations?

Are all $\Lambda^\mu{}_\nu$ allowed? If $\underbrace{0 = \eta_{\mu\nu} dx^\mu dx^\nu}$

$0 = \eta_{\mu\nu} (\Lambda^\mu{}_\rho dx^\rho)(\Lambda^\nu{}_\sigma dx^\sigma)$ ← ensure Λ doesn't change c.

$= dx^T \underbrace{\Lambda^T \eta \Lambda} dx$

for any vector $a^\mu \to \Lambda^\mu{}_\nu a^\nu$

$a_\mu \to \Lambda^\nu{}_\mu a_\nu$

$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}$

infinitesimal

Helpful: $\Lambda^\rho{}_\mu = \delta^\rho{}_\mu + \varepsilon^\rho{}_\mu$

$\eta_{\mu\nu} = (\delta+\varepsilon)^\rho{}_\mu (\delta+\varepsilon)^\sigma{}_\nu \eta_{\rho\sigma}$

$= \underbrace{\delta^\rho{}_\mu \delta^\sigma{}_\nu \eta_{\rho\sigma}}_{\eta_{\mu\nu}} + \eta_{\mu\sigma} \varepsilon^\sigma{}_\nu + \varepsilon^\rho{}_\mu \eta_{\rho\nu}$

$\boxed{0 = \eta_{\mu\sigma} \varepsilon^\sigma{}_\nu + \varepsilon^\rho{}_\mu \eta_{\rho\nu}}$

1) Try: $\mu = \nu = x$

$0 = 1 \cdot \varepsilon^x{}_x + 1 \cdot \varepsilon^x{}_x$

$\varepsilon^x{}_x = 0.$

2) Try: $\mu = x,\ \nu = y$

$\varepsilon^x{}_y + \varepsilon^y{}_x = 0$

3) Try: $\mu = t,\ \nu = x$;

$0 = -\varepsilon^t{}_x + \varepsilon^x{}_t.$

3 Summarize the global and local perspectives on relativistic symmetry.

1) Global [useful when doing effective theory]

$x^\mu \to x^\mu + a^\mu$ and $x^\mu \to \Lambda^\mu{}_\nu x^\nu$

[$t=0$ unimportant]

$\underline{\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}}$

c is same for all

Observe: $a_\mu b^\mu$ is Λ-invariant:

$$\eta_{\mu\nu} a^\nu b^\mu = \eta_{\mu\nu}(\Lambda^\nu{}_\rho a^\rho)(\Lambda^\mu{}_\sigma b^\sigma)$$
$$= \underbrace{(\Lambda^\nu{}_\rho \Lambda^\mu{}_\sigma \eta_{\mu\nu})}_{=\eta_{\rho\sigma}} a^\rho b^\sigma = \eta_{\sigma\rho} a^\rho b^\sigma$$

2) Local: [Noether's Thm]

$\Lambda = (\delta + \varepsilon_1)(\delta + \varepsilon_2)(\delta + \varepsilon_3)\dots$

$x \to x + \beta_x \cdot ct$

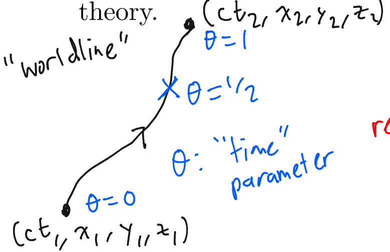
$ct \to ct + \beta_x \cdot x$

boost

$$\underbrace{\Lambda^\mu{}_\nu}_{\delta+\varepsilon} = \begin{pmatrix} 1 & \beta_x & \beta_y & \beta_z \\ \beta_x & 1 & \varepsilon_{xy} & \varepsilon_{xz} \\ \beta_y & -\varepsilon_{xy} & 1 & \varepsilon_{yz} \\ \beta_z & -\varepsilon_{xz} & -\varepsilon_{yz} & 1 \end{pmatrix}$$

rot.

4 Deduce the action for a free relativistic particle based on effective theory.



Build:

$$S = \int d\theta \, L\left(x^\mu, \frac{dx^\mu}{d\theta}, \ldots\right)$$

Because θ is not physical:

$\tilde{\theta} = f(\theta)$ [f invertible]

$\tilde{\theta}$ is just as valid

restrict to

$f(\theta) = a \cdot \theta$

$$S = \int (a \cdot d\theta) \, L\left(x^\mu, \frac{1}{a}\frac{dx^\mu}{d\theta}, \ldots\right)$$

$x^\mu \to x^\mu + a^\mu$

$L(x, \ldots) = L(x+a, \ldots) = L(\ldots)$

Λ-invariance: contracted indices

So $L\left(\frac{dx_\mu}{d\theta}\frac{dx^\mu}{d\theta}, \ldots\right)$

Fix: $$S = \int d\theta \sqrt{-\frac{dx^\mu}{d\theta}\frac{dx_\mu}{d\theta}} \cdot (-mc)$$

5 Discuss the non-relativistic limit of motion.

Can make $\theta = t$

$$S = -mc\int dt\sqrt{c^2\left(\frac{dt}{dt}\right)^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}$$

$$= -mc^2\int dt\sqrt{1 - \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}$$

mass

sign in $\sqrt{\ }$ needed so particles follow timelike trajectory

non-relativistic limit: $c \to \infty$

$$S = -mc^2\int dt\left[1 - \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}\right]$$

$$\simeq \frac{m}{2}\int dt\left[\dot{x}^2+\dot{y}^2+\dot{z}^2\right]$$