PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 30

Integrable systems

October 31

1 What are the conditions for a Hamiltonian system to be integrable?

integrable = NOT chaotic.

↖ "deterministic + unpredictable"

heuristic: integrable = exactly solvable?

in Hamiltonian systems, have $2n$-dim phase space...

integrable = has n functions $\{J_1, \dots, J_n\}$ such that

① $[H, J_i] = 0$; ② $[J_i, J_j] = 0$; ③ J_i's "independent" (e.g. $J_2 \neq f(J_1^2)$)

$\boxed{n \text{ constants of motion} = \text{integrable}}$

- Locally, try to build J's?
- To have integrability, J's must be defined globally.
- Most systems = chaos!

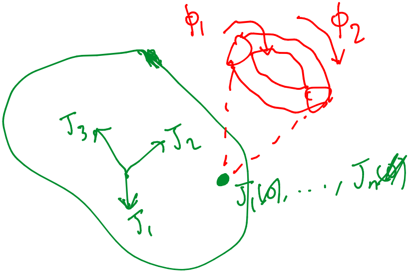
2 Describe phase space in terms of the invariant tori.

Assume: H and J_i's are t-independent. (Choose $J_1 = H \ldots$)

Assume: for some energy (H) range: sets of const. E are compact.

phase space:

Remaining n-dimensions are torus T^n (Liouville-Arnold Thm).



Phase space coords $(J_1, \ldots, J_n ; \phi_1, \ldots, \phi_n)$ are action-angle variables.

$$\dot{\phi}_j = \frac{\partial H}{\partial J_j} = \omega_j ; \quad \dot{J}_j = -\frac{\partial H}{\partial \phi_j} = 0.$$

So $H(J_1, \ldots, J_n)$.

$$\phi_j(t) = \phi_j(0) + \omega_j t$$

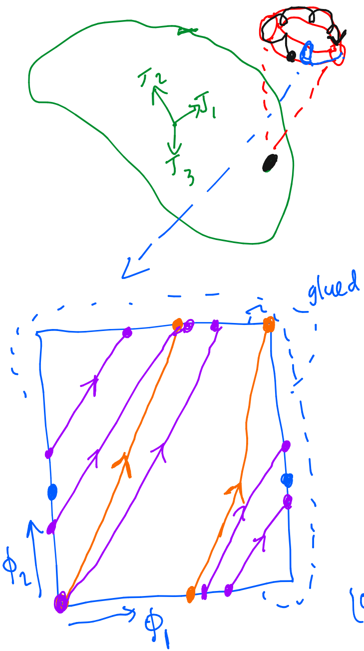
const.

$$\frac{dJ_i}{dt} = [J_i, H] = 0.$$

3 Describe periodic vs. quasiperiodic motion on the tori.

$$\dot{J}_i = 0 \qquad i = 1, \dots, n$$

$$\underline{\dot{\phi}_j = \omega_j} = \frac{\partial H}{\partial J_j} = \text{const.}$$



- Periodic (commensurate):
  $\underline{\omega_2 = 2\omega_1}$

  In general: $\left.\frac{\omega_2}{\omega_1} = \frac{p}{q}\right\}$ rational

- generic: incommensurate (in = not)
  $\frac{\omega_2}{\omega_1}$ = irrational. e.g $\omega_2 = \sqrt{2}\,\omega_1$

  – as $t \to \infty$, trajectory will "uniformly" traverse torus.

$$(\phi_1, \phi_2, \dots, \phi_n) \sim (\phi_1 + 2\pi, \dots, \phi_n) \sim \dots (\phi_1, \dots, \phi_n + 2\pi)$$

4 Describe the general dynamics of a system in terms of a Fourier series in the action-angle variables.

Observable $F = F(\underbrace{J_1, \ldots, J_n}_{\text{const.}}, \underbrace{\phi_1, \ldots, \phi_n}_{\phi_j \sim \phi_j + 2\pi \text{ (periodic)}})$

$$F = \sum_{m_1, \ldots, m_n \in \mathbb{Z}} F_{m_1 \cdots m_n}(J_1, \ldots, J_n) e^{im_1\phi_1 + im_2\phi_2 + \cdots + im_n\phi_n}$$

NOTE: did NOT write $\cancel{m_i\phi_i} = m_1\phi_1 + \cdots + m_n\phi_n$
(for AA: write sums explicitly...)

$$F(t) = \sum_{m_1, \ldots, m_n} F_{m_1 \cdots m_n}(J_1, \ldots, J_n)\, e^{im_1\phi_1(0) + \cdots + im_n\phi_n(0)} \cdot e^{i(m_1\omega_1 + \cdots + m_n\omega_n)t}$$