PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 31

Perturbation theory: one degree of freedom

November 2

1 Describe qualitatively the motion of a perturbed harmonic oscillator.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \underbrace{bx^4}_{\text{perturbation}}$$

↓

small dimensionless quantity

Solve order-by-order in small parameter (b)

$$H = \underbrace{\left[\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2\right]}_{H_0} + \underbrace{bx^4}_{\varepsilon H_1}$$

$$\boxed{H = H_0 + \varepsilon H_1}$$

small ⟷ ε = "book-keeping device"

$\varepsilon \to 1$ $\quad (b \to \varepsilon b)$

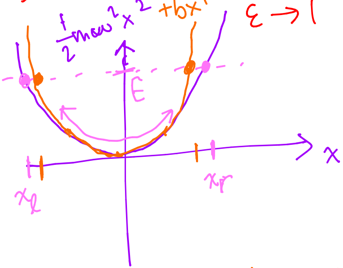
$bx^4 \ll m\omega^2 x^2$

Lazy $O(1)$ constants:

$$b\left(\frac{E}{m\omega^2}\right)^2 \ll E$$

$$\boxed{b \ll \frac{(m\omega^2)^2}{E}}$$

dim-less: $\left.\frac{bE}{(m\omega^2)^2}\right\} \varepsilon$?



$(b>0)$

expect: bounded, periodic motion

2 Solve the problem with (naive) perturbation theory.

Try: $H = H_0 + \varepsilon H_1$

$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2, \quad H_1 = b x^4$

$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$

$p(t) = p_0(t) + \varepsilon p_1(t) + \cdots$

Plug & chug:

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{x}_0 + \varepsilon \dot{x}_1 + \cdots = \left.\frac{\partial H_0}{\partial p}\right|_{p_0+\varepsilon p_1+\cdots} + \cancel{\varepsilon \left.\frac{\partial H_1}{\partial p}\right|_{p_0}} + \cdots$$

$$= \frac{p}{m} = \frac{p_0 + \varepsilon p_1 + \cdots}{m}$$

$$\left.\begin{aligned}\dot{x}_0 &= p_0/m \\ \dot{x}_1 &= p_1/m \quad \cdots\end{aligned}\right\} \quad \dot{x} = \frac{p}{m}$$

$$\dot{p}_0 + \varepsilon \dot{p}_1 + \cdots = -\frac{\partial (H_0 + \varepsilon H_1)}{\partial x}$$

$$= -m\omega^2 (x_0 + \varepsilon x_1 + \cdots) - \varepsilon \cdot 4b (x_0 + \varepsilon x_1 + \cdots)^3$$

$$\mathcal{O}(\varepsilon^0): \quad \left.\begin{aligned}\dot{p}_0 &= -m\omega^2 x_0 \\ \ddot{x}_0 &= -\omega^2 x_0\end{aligned}\right\} x_0 = a\cos(\omega t)$$

$\omega + \varepsilon\omega_1$

$$\mathcal{O}(\varepsilon^1): \quad m\ddot{x}_1 = -m\omega^2 x_1 - 4b x_0^3$$

Solution: $$x_1(t) = -\frac{A^3}{2\omega^2}\left[\boxed{3\omega t \sin(\omega t)} + \frac{\cos(\omega t)}{4} - \frac{\cos(3\omega t)}{4}\right]$$

blows up !!

$\cos((\omega + \varepsilon\omega_1)t)$

$\boxed{3}$ Why is it better to do perturbation theory with action-angle variables?

PT can be better behaved in other coords (not x, p)

Action-angle variables are better.

- leading order is simple.
- AA should exist for both $\underbrace{H_0}_{(J_0, \phi_0)}$ AND $\underbrace{H_0 + \varepsilon H_1}_{(J, \phi)}$.

$(J_0, \phi_0) \longrightarrow (J, \phi)$

CT perturbatively?

Caution: "hard" at higher order: see JS.

Goal: $H(J) = H_0(J, \phi) + \varepsilon H_1(J, \phi)$

$$\downarrow$$

$$\left.\begin{aligned} J &= J_0 + \varepsilon J_1 + \cdots \\ \phi &= \phi_0 + \varepsilon \phi_1 + \cdots \end{aligned}\right\}$$

Look for type 2 generating fcn: $S(\phi_0, J)$:

$$J_0 = \frac{\partial S}{\partial \phi_0} = J + O(\varepsilon)$$

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + O(\varepsilon)$$

$$S = \phi_0 J + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots$$

4 What is the perturbed Hamiltonian, to first order?

$$H_0(J_0) + \varepsilon \underbrace{H_1(J_0,\phi_0)}_{\text{computed}} = H(J) = H(J_0 + \varepsilon J_1 + \cdots)$$

dif. btwn J, J_0 is $O(\varepsilon)$

$$H(J) = H_0(J - \varepsilon J_1 + \cdots) + \varepsilon H_1(J, \phi_0)$$

$$= H_0(J) - \varepsilon \frac{\partial H_0}{\partial J_0} J_1 + \varepsilon H_1$$

$$J_0 = \frac{\partial S}{\partial \phi_0} = J + \varepsilon \frac{\partial S_1}{\partial \phi_0} + \cdots = J - \varepsilon J_1 + \cdots, \quad \text{so } J_1 = -\frac{\partial S_1}{\partial \phi_0}$$

$$H(J) = H_0(J) + \varepsilon \underbrace{\left[H_1 + \frac{\partial H_0}{\partial J_0} \frac{\partial S_1}{\partial \phi_0} \right]}$$

Goal:
- only depend on J!
- b/c S_1 is unknown, use S_1 to cancel ϕ_0 dep. in H_1!

$$H_1 = \sum_{m \in \mathbb{Z}} e^{im\phi_0} \cdot h_m(J)$$

$$S_1 = \sum_{m \in \mathbb{Z}} e^{im\phi_0} s_m(J)$$

$$\sum_{m \in \mathbb{Z}} e^{im\phi_0} \left[h_m + \underbrace{\left(\frac{\partial H_0}{\partial J_0}\right)}_{\omega_0} im\, s_m \right]$$

$m = 0$: no ϕ_0-dep!

$$H = H_0 + \varepsilon h_0$$

$m \neq 0$:

$$s_m = -\frac{h_m}{im\,\omega_0}$$

5 Solve the perturbed harmonic oscillator.

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \quad ; \quad H_1 = bx^4$$

$\downarrow$ cf Lec 29?

$$H_0 = \omega J_0$$

$$x = \sqrt{\frac{2E}{m\omega^2}}\cos\phi_0 = \sqrt{\frac{2J_0}{m\omega}}\cos\phi_0$$

$$p = \sqrt{2J_0 m\omega}\,(-\sin\phi_0)$$

Write $H_1(J_0,\phi_0) = b\left(\frac{2J_0}{m\omega}\right)^2\cos^4\phi_0$

$$= \frac{4bJ_0^2}{m^2\omega^2}\left[\frac{3}{8} + \frac{\cos(4\phi_0)}{8} + \frac{\cos(2\phi_0)}{2}\right]$$

$$h_0 = \frac{1}{2\pi}\int_0^{2\pi} d\phi_0\, H_1 = \frac{3}{2}\frac{bJ_0^2}{m^2\omega^2}$$

$$H(J) = \omega J + \varepsilon h_0(J) \quad J \approx J_0$$

$$= \omega J + \frac{3}{2}\varepsilon\frac{bJ^2}{m^2\omega^2}$$

What's oscillation frequency? $J \approx E/\omega$

$$\omega_{new}(J) = \frac{\partial H(J)}{\partial J} = \omega + 3\varepsilon\cdot\frac{bJ}{m^2\omega^2} = \omega\left[1 + 3\varepsilon\underbrace{\frac{bE}{(m\omega^2)^2}}_{\text{well-behaved if } \ll 1}\right]$$