PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 32

Perturbation theory: many degrees of freedom

November 4

1 Set up perturbation theory in action-angle variables for multiple degrees of freedom.

P.T. w/ A.A variables:

$$\underbrace{H_0(J_0^\alpha) + \varepsilon H_1(J_0^\alpha, \phi_0^\alpha)}\xrightarrow{\text{Goal: CT}} H(J^\alpha)$$

$\alpha = 1, \ldots, n$

AA: $(J_0^\alpha, \phi_0^\alpha)$ ← old, integrable

$[\phi_0^\alpha, J_0^\beta] = \delta^{\alpha\beta}$

new AAs:

new → (J^α, ϕ^α)

$[\phi^\alpha, J^\beta] = \delta^{\alpha\beta}$.

CT can be found as Taylor series in ε ("perturbatively")

$$J^\alpha = J_0^\alpha + \varepsilon J_1^\alpha + \cdots$$

$$\phi^\alpha = \phi_0^\alpha + \varepsilon \phi_1^\alpha + \cdots$$

Type 2 CT: $(\phi_0^\alpha, J^\alpha)$ ind.

$$S(\phi_0^\alpha, J^\alpha) = \sum_\alpha \phi_0^\alpha J^\alpha + \varepsilon S_1 + \cdots$$

$$J_0^\alpha = \frac{\partial S}{\partial \phi_0^\alpha} = J^\alpha + \mathcal{O}(\varepsilon) \;;\quad \phi^\alpha = \frac{\partial S}{\partial J^\alpha} = \phi_0^\alpha + \mathcal{O}(\varepsilon)$$

2 What is the first order Hamiltonian?

$$H(J^\alpha) = H_0(J_0^\alpha) + \varepsilon H_1(J_0^\alpha, \phi_0^\alpha)$$

$$S = \sum_\alpha \phi_0^\alpha J^\alpha + \varepsilon S_1 + \cdots$$

$$J_0^\alpha = J^\alpha - \varepsilon J_1^\alpha + \cdots = \frac{\partial S}{\partial \phi_0^\alpha} = J^\alpha + \varepsilon \frac{\partial S_1(J,\phi_0)}{\partial \phi_0^\alpha} + \cdots$$

Up to $\mathcal{O}(\varepsilon)$:

$$H(J^\alpha) = H_0\left(J^\alpha + \varepsilon \frac{\partial S_1}{\partial \phi_0^\alpha}\right) + \varepsilon H_1(J^\alpha, \phi_0^\alpha) + \mathcal{O}(\varepsilon^2)$$

$$H(J^\alpha) = H_0(J^\alpha) + \sum_\beta \frac{\partial H_0}{\partial J^\beta} \varepsilon \frac{\partial S_1}{\partial \phi_0^\beta} + \varepsilon H_1(J, \phi_0)$$

$$H_1 = \sum h_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}_0}$$

$$\vec{m} = (m_1, \ldots, m_n)$$

$$\vec{m}\cdot\vec{\phi}_0 = \sum_\alpha m^\alpha \phi_0^\alpha \;;$$

$$S_1 = \sum s_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}_0}$$

$$\rightarrow \; = H_0(J^\alpha) + \sum_{\vec{m}} \varepsilon \Big[\underbrace{h_{\vec{m}} + \sum_\beta \frac{\partial H_0}{\partial J^\beta} i m^\beta}\Big] e^{i\vec{m}\cdot\vec{\phi}_0}$$

$s_{\vec{m}}$

$$\omega^\beta = \left.\frac{\partial H_0}{\partial J^\beta}\right|_J$$

if $m_1 = m_2 \cdots = m_n = 0$:

$$H(J) = H_0(J) + \varepsilon h_{\vec{0}}(J)$$

else: $\boxed{0 = h_{\vec{m}} + i(\vec{m}\cdot\vec{\omega}) s_{\vec{m}}}$

3 When will the approach fail?

$(\vec{m} \neq \vec{0})$

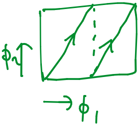
$$S_{\vec{m}} = -\frac{h_{\vec{m}}}{i\,\vec{m}\cdot\vec{\omega}}$$ be "small"?

CT to new AA

If denominator small, PT fails?

Suppose 2 DOF.

Case 1: commensurate:



$$\frac{\omega^1}{\omega^2} = p/q \in \text{integers}$$

$$m^1\omega^1 + m^2\omega^2 = 0$$

$$m^2 = p, \quad m^1 = -q$$

$$-\cancel{q}\left(\frac{p}{\cancel{q}}\omega^2\right) + p\,\omega^2 = 0.$$

If $h_{\vec{m}} \neq 0$, PT failed.

Case 2: incommensurate: $\frac{\omega^1}{\omega^2}$ = irrational. $\vec{m}\cdot\vec{\omega} \neq 0$.

close to rational #s..

$$S = \sum_\beta \phi_0^\beta J^\beta + \sum -\frac{\varepsilon h_{\vec{m}}}{i\,\vec{m}\cdot\vec{\omega}} e^{i\vec{m}\cdot\vec{\Phi}_0}$$

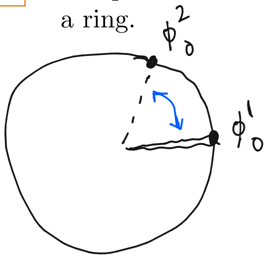
small.

if $h_{\vec{m}}$ fall off sufficiently fast

.. PT OK.

KAM Thm

4 Use perturbation theory to describe the interaction of two particles on a ring.



$$H_0 = \frac{(J_0^1)^2}{2I} + \frac{(J_0^2)^2}{2I} + \varepsilon \cdot b\cos(\phi_0^1 - \phi_0^2)$$

($J_0^{1,2}$ ang. mom.)

First order PT:

$$H = H_0 + \varepsilon \underbrace{h_0}$$

$$\int \frac{d\phi_0^1}{2\pi} \int \frac{d\phi_0^2}{2\pi}\, b\cos(\ ) = 0$$

$$H(J) \approx H_0(J)$$

Did PT fail?

$$\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\varepsilon H_1 = \varepsilon \frac{b}{2}\left[e^{i(\phi_0^1 - \phi_0^2)} + e^{-i(\phi_0^1 - \phi_0^2)}\right]$$

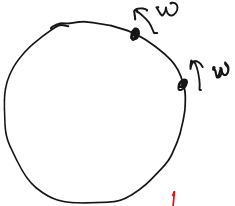
$\hookrightarrow$ $h_{1,-1} = h_{-1,1} = b/2$, rest $= 0$.

$$S_{1,-1} = -\frac{b/2}{i(\omega^1 - \omega^2)}.$$

PT fail if $\omega^1 = \omega^2$

$$\omega^1 = \frac{\partial H_0}{\partial J_0^1} = \frac{J_0^1}{I} \qquad \omega^2 = \frac{J_0^2}{I}$$

5 Why does perturbation theory fail if $J_0^1 = J_0^2$?



$\bar{J}_0 = J_0^1 + J_0^2$; $\bar{\phi}_0 = \frac{\phi_0^1 + \phi_0^2}{2}$

$\tilde{J}_0 = \frac{J_0^1 - J_0^2}{2}$; $\tilde{\phi}_0 = \phi_0^1 - \phi_0^2$

$[\bar{J}_0, \tilde{\phi}_0] = 0$, e.g.

$$H = \left(\frac{\bar{J}_0^2}{4I}\right) + \left(\frac{\tilde{J}_0^2}{I} + \varepsilon b \cos\tilde{\phi}_0\right)$$

PT failed if: $J_0^1 = J_0^2$. $\tilde{J}_0 = 0$.

$t=0$, w/ $\tilde{J}_0 = 0$.