PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 33

The adiabatic theorem

November 7

1 State the adiabatic theorem.

Give $H(J, \phi\, ; \lambda)$

J: A.A. (not in AA) — ϕ; λ: parameter - not coord - vary w/ time

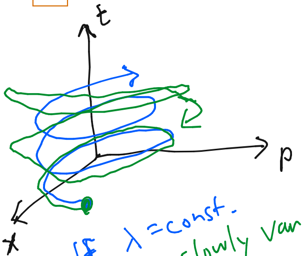
Ex: $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$; $\lambda \to \omega$

Adiabatic Thm: If we vary λ slowly, then $\dot{J} \approx 0$ [ie J const.]

J ← adiabatic invariant.

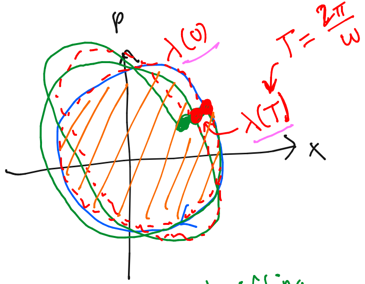
How slowly: $\frac{\dot{\lambda}}{\lambda} \ll \omega = \frac{\partial H}{\partial J}$

2 Prove the adiabatic theorem.



If $\lambda = \text{const.}$

If λ slowly varying



orbit <u>slowly</u> drifting

<u>Key:</u> Time evolution in Ham is a canonical transf.

$$F(x(0), p(0)) \to (x(T), p(T))$$

F = canonical,

$$J = \oint dx\, p = \text{Area(orbit)}$$

$\downarrow$ Stokes' Thm

$$\int dx\, dp \cdot \cancel{\omega_{xp}}^{\,1}$$

CT generated by time evol.

$$\int dx\, dp\ \omega_{xp} \to \int dx(T)\, dp(T)$$

$$\underline{\omega_{x(T)\,p(T)}} = 1$$

= Area(orbit at $t = T$)

[Liouville's Thm: area unchanged]

3 What happens to a harmonic oscillator whose spring constant slowly increases?

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega(t)^2 x^2$$

✓

Adiabatic? $\frac{\dot{\omega}}{\omega} \ll \frac{\partial H}{\partial J} = \frac{2\pi}{T} = \omega$, so $\dot{\omega} \ll \omega^2$

energy

$J = \text{Area(orbit)} = \frac{E}{\omega} \approx$ const.

$\omega(0) = \omega_0 \longrightarrow \omega(t\to\infty) = 10\omega_0$. How much does amplitude of oscillations change?

$X =$ amplitude of osc.

$E = \frac{1}{2} m\omega^2 X^2$

$J = \frac{E}{\omega} = \frac{m}{2}\omega X^2$

$X = \sqrt{\frac{2J}{m\omega}}$

$X \propto \omega^{-1/2}$.

$\frac{X(\infty)}{X(0)} = \frac{1}{\sqrt{10}}$.

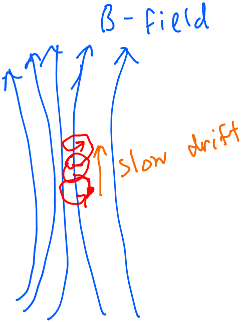
4 Describe the motion of a charged particle in a magnetic field.

$$H = \frac{(p_i - qA_i)(p_i - qA_i)}{2m}$$

[sum convention on repeated index]

Periodicity to cyclotron orbit;

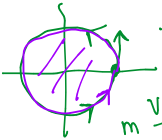
$$J = \oint d\vec{x}_\perp \cdot \vec{p}_\perp = \oint d\vec{x}_\perp \cdot (\underbrace{m\vec{v}_\perp}_{} + \underbrace{q\vec{A}_\perp}_{})$$



$$\int dt\, m\left(\frac{qB}{m} r\right)^2 = \frac{2\pi}{\cancel{\omega_c}} \cdot \cancel{\frac{qB}{m}}\, qBr^2 = 2\pi q B r^2$$

$$\oint q\, d\vec{x}_\perp \cdot \vec{A}_\perp = -q \int_{\text{loop}} d\vec{a} \cdot \vec{B} = -qB \cdot \pi r^2.$$

"cyclotron"



$$dx_\perp \cdot v_\perp = v^2\, dt$$

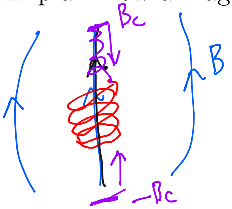
$$m\frac{v^2}{r} = qvB \hookrightarrow mv = qBr$$

$$v = \underbrace{\frac{qB}{m}}_{=\omega_c} r$$

$J \propto$ magnetic flux thru orbit

$$J \propto Br^2 \propto \frac{v_\perp^2}{B}$$

5 Explain how a magnetic mirror works.



Adiabatic invar:

$$\tilde{J} = \frac{v_\perp^2}{B_\parallel}$$

How much kinetic energy? $E = \frac{m}{2}\left(v_\perp^2 + v_\parallel^2\right)$

- since $q\vec{v}\times\vec{B}$ does no work, E = conserved energy
[also adiabatic!]

$$E = \frac{m}{2}\left[B_\parallel \tilde{J} + v_\parallel^2\right]$$

Particle cannot proceed past $B_\parallel > \frac{2E}{m\tilde{J}} = B_c$