PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 34

The Hénon-Heiles Hamiltonian

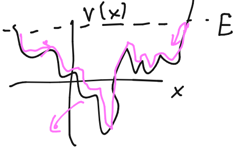
November 9

1 What is the Hénon-Heiles Hamiltonian?

Claim: $H(x_i, p_i)$ is independent of t.

How many DOF for chaos?

Need $\underline{2 \text{ DOF}}$. ($H$ is const.)



(J, ϕ) always definable.

Hénon-Heiles Hamiltonian:

$$H = \underbrace{\frac{p^2+q^2}{2m} + k\frac{x^2+y^2}{2}}_{H_0} + \underbrace{k\cdot a \underbrace{\left(x^2 y - \frac{y^3}{3}\right)}_{\frac{1}{3}r^3\sin(3\theta)}}_{\varepsilon H_1} + \cancel{\dots (r^4)}$$

$(p_x \to p,\ p_y \to q)$

As $y \to +\infty$, $-\frac{y^3}{3} \to -\infty$.

Avoid running to ∞:

$H(p=q=x=0,\ y) \le ?$

$= \frac{y^2}{2} - \frac{y^3}{3}$

Look for maximum:

$\frac{dH}{dy} = 0 = y - y^2$, so $y = 1$

$H = E \le 1/6$.

$\dot p = -\frac{\partial H}{\partial x} = -x - 2xy$; $\dot q = -y - x^2 + y^2$; $\dot x = p$ $\quad \dot y = q$

2 For small $E \approx 0.058$, the system is integrable: $\{y(t), q(t)\}$

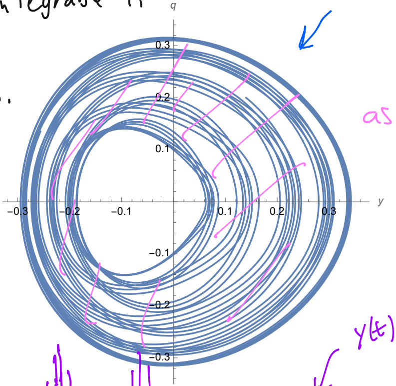
$q(t) = \dot{y}$

$t \sim 200.$

Recall: system is integrable if $H(J_1, J_2)$ is expressed in AA vars.

$$\dot{J}_1 = 0$$
$$\dot{J}_2 = 0$$
$$\dot{\phi}_1 = \frac{\partial H}{\partial J_1} = \text{const.}$$
$$\dot{\phi}_2 = \frac{\partial H}{\partial J_2} = \text{const.}$$

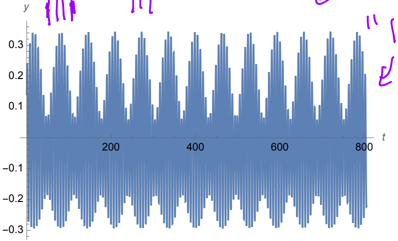
quasiperiodic.



as $t \to \infty$, fill in more of plane...

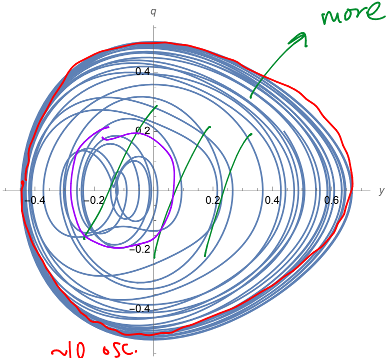
$y(t)$

"looks quasi-per."

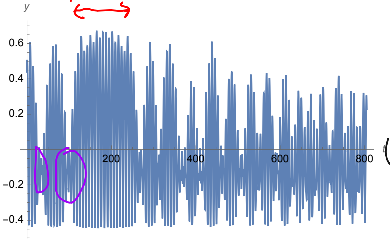


$\boxed{3}$ But at larger $E \approx 0.13$, chaos can ensue!

more of phase space filled in.



~10 osc.



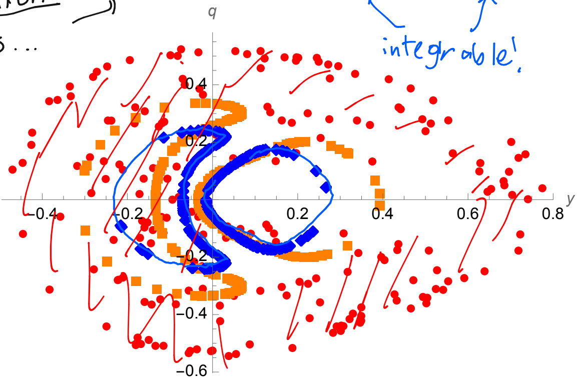
← erratic, no (quasi)periodicity?

4 Increasing E for the same initial conditions, we can see the onset of chaos.

3 diff. sims: $E \approx 0.03$ $E \approx 0.06$ $E \approx 0.15$

<u>Poincaré section</u>

- integrate EOMs . . .
- record times when $x(t_i) = 0$ $t_1, t_2, t_3, \ldots$



integrable!

chaos.

fill in frac 2d subspace

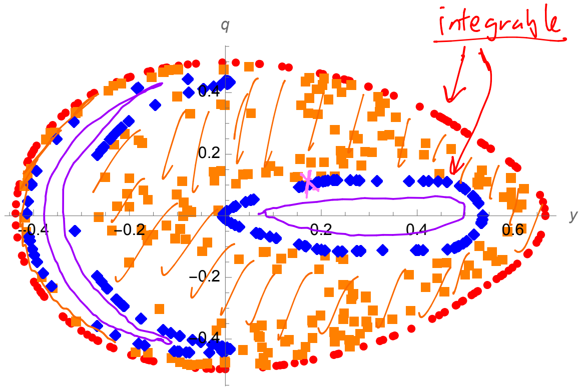
In AA coords, integrable system → quasi-periodic on "invariant torus".

E = conserved; restrict to 3-dim subspace

intersection of 3-dim subspaces will be 2-dim.

phase space has 4 coords, but dynamics explores only intersect. of 3d with 2d → 1d.

5 At the same $E = 0.125$, we can find initial conditions with either chaos or integrability.



chaos "filling gaps" between "islands of integrability"