PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 35

The kicked rotor

November 11

1 What is the kicked rotor?

Last time: $H(x_i, p_i)$: 2 DOF for chaos. (E conserved ↳ AA vars for 1 DOF)

Today: $H(J, \phi, t)$: can have chaos w/ 1 DOF! [E not conserved]

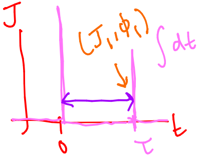
Dirac δ

$$H(J,\phi,t) = \frac{J^2}{2I} + \sum_{n\in\mathbb{Z}} \delta(t-n\tau)\cdot \varepsilon_0 \cos\phi$$

Write down EOM:

$$\dot\phi = \frac{\partial H}{\partial J} = \frac{J}{I} \leftarrow$$

$$\dot J = -\frac{\partial H}{\partial \phi} = \varepsilon_0 \sin\phi \sum_{n\in\mathbb{Z}} \delta(t-n\tau)$$



$\int dt\, \dot J \sim \varepsilon_0 \sin\phi$

Start at $t=0^+$ — right after 0.

$\phi(0^+) = \phi_0$; $J(0^+) = J_1$

$\dot J = 0$, for $0<t<\tau$:

$J(t) = J_1$ $\qquad J(\tau^-) = J_1$ — just before τ

$\dot\phi = \frac{J}{I}$,

$$\phi(t) = \phi_0 + \frac{J_1}{I} t$$

$$\phi(\tau^-) = \phi_0 + J_1/I\,\tau = \phi_1$$

$\boxed{2}$ Reduce the dynamics to a discrete map.

So far: btwn $0 < t < \tau$: $J = \text{const.}$ $\quad J(\tau^-) = J_1$

$$\phi(\tau^-) = \phi_1 = \phi_0 + \tau/I \, J_1$$

small

$$\tau^- \mapsto \tau - \Delta \qquad \tau^+ \mapsto \tau + \Delta$$

$$\phi(\tau+\Delta) = \phi(\tau-\Delta) + \int_{\tau-\Delta}^{\tau+\Delta} dt\,\dot\phi \approx \phi(\tau-\Delta) + \left(\int_{\tau-\Delta}^{\tau} + \int_{\tau}^{\tau+\Delta}\right)$$

$$\cancel{+\Delta\left(\frac{J_1}{I} + \frac{J_2}{I}\right)} \quad J(\tau^+)$$

$$\overset{J_2}{J(\tau+\Delta)} = \overset{J_1}{J(\tau-\Delta)} + \int_{\tau-\Delta}^{\tau+\Delta} dt\,\dot J = J(\tau-\Delta) + \underbrace{\int dt\, \varepsilon_0 \sin\phi\, \delta(t-\tau)}$$

$$= \underbrace{\varepsilon_0 \sin\phi(t=\tau)}_{\text{no } \frac{1}{\Delta}}$$

Limit $\Delta \to 0$: $\phi(t)$ continuous.

$$\phi_1 = \phi_0 + \tau/I \, J_1$$

$$J_2 = J_1 + \varepsilon_0 \sin\phi_1$$

Iterate: $J_n = J(t = n\tau^-)$, $\phi_n = \phi(t = n\tau^-)$

$$J_{n+1} = J_n + \varepsilon_0 \sin\phi_n$$

$$\phi_{n+1} = \phi_n + \tau/I \, J_{n+1}$$

3 Non-dimensionalize the problem.

[quant] = units/dimensions.

$\boxed{\text{3 parameters:}}$ τ, I, ε_0.

$$\left.\begin{aligned} [\phi] &= 1 \\ [J] &= [M][L]^2[T]^{-1} \\ [t] &= [T] \end{aligned}\right\} \text{measure in "dim. less" units}$$

length, mass

$$\tilde{t} = t/\tau$$

e.g. $\phi_n = \phi(\tilde{t} = n)$

$$\tilde{J} = J \cdot \frac{\tau}{I}$$

$$\varepsilon = \varepsilon_0 \tau / I = \text{dim. less}$$

$[\tau] = [T]$

$[I] = [M][L]^2$

$[\varepsilon_0] = [T] \cdot [\text{energy}] = [J]$

$H = \varepsilon_0 \delta(t)$ $\quad [\delta(t)] = [T]^{-1}$

Using these units:

$$\left.\begin{aligned} \tilde{J}_{n+1} &= \tilde{J}_n + \varepsilon \sin \phi_n \\ \phi_{n+1} &= \phi_n + \tilde{J}_{n+1} \end{aligned}\right\} \text{kicked rotor map.}$$

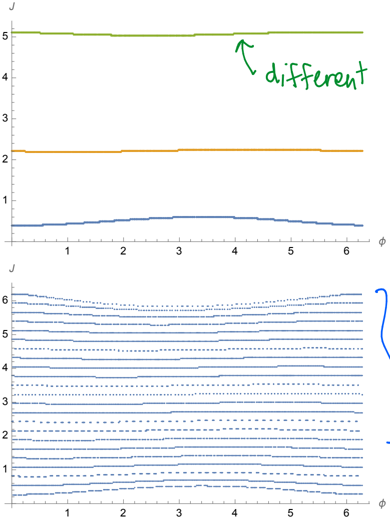
ϕ is ang coord. $\phi_n \sim \phi_n + 2\pi$

ALSO: $\tilde{J}_n \sim \tilde{J}_n + 2\pi$.

$\boxed{4}$ Plots at $\epsilon = 0.05$: $\leftarrow$ kicks weak

integrable

"emergent conserved J_{AA}"



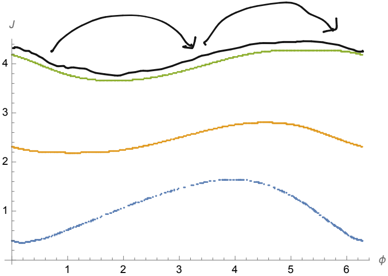
different (ϕ_0, J_0)

~20 ICs.

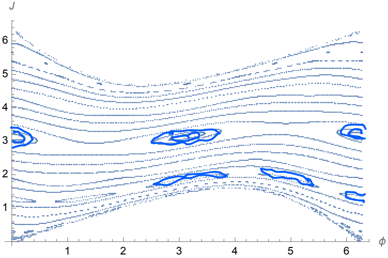
5 Plots at $\epsilon = 0.6$:

trajectories only explore 1d ...

as if function $f(J_n, \phi_n) = \text{const.}$ for each trajectory



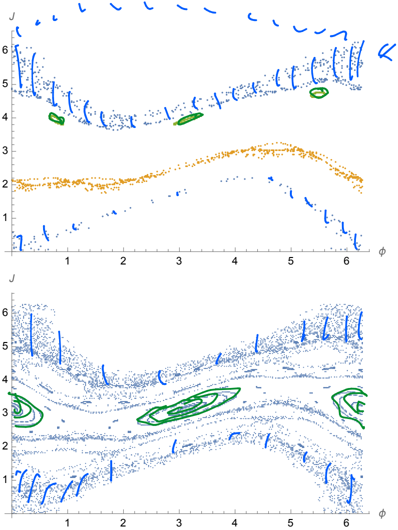
kicks halfway $\phi \to \phi + \pi$ each time.



6 Plots at $\epsilon = 1.05$:

Onset of chaos

$\varepsilon_c \sim 0.95$.

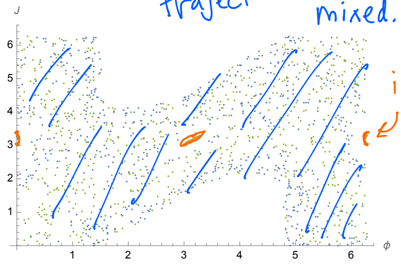


trajectories explores finite fraction of phase space $\rightarrow$ chaos.

"islands of integrability"

7 Plots at $\epsilon = 1.8$:

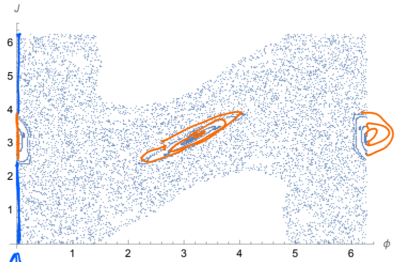
trajectories w/ very diff. ICs.
mixed... chaos



integrable island

chaos

integ.



IC have $\phi \approx 0$