PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 36

The logistic map

November 14

1 What is the logistic map?

Hamiltonian dynamics → kicked rotor → logistic map

Hamiltonian dynamics: (conserved H) (Hénon–Heiles), 4-dim phase space

Kicked rotor: ($H \neq$ conserved), discrete map (2 var)

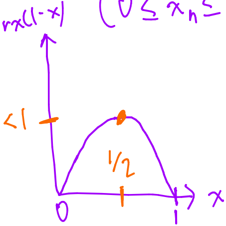
minimal model of chaos: logistic map (non-Hamiltonian) (dissipative)

Logistic map: $\{x_0, x_1, x_2, \ldots\}$ with $0 \le x_n \le 1$ and

$(r > 0)$ $\quad x_{n+1} = r x_n (1 - x_n)$ (inspired by population dynamics)

Want x_n's bounded. $|x_n|$ very large? $x_{n+1} \propto -x_n^2 \longrightarrow -\infty$ (at large n)

Want $x_n \in [0, 1]$. $(0 \le x_n \le 1)$ Need $0 \le r x_n (1 - x_n) \le 1$.



$r \cdot \frac{1}{2}\left(1 - \frac{1}{2}\right) \le 1$

$\frac{r}{4} \le 1$

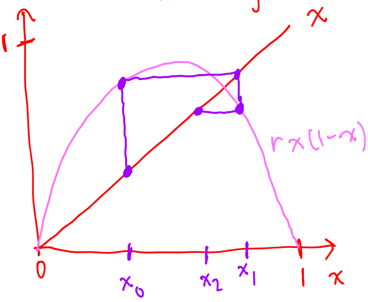
$\boxed{r \le 4}$

Dissipative: $x_n \to$ subset of $[0, 1]$ (phase space)

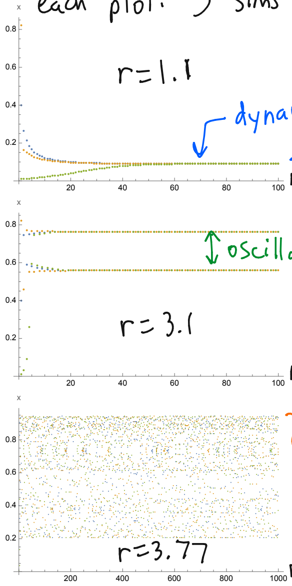
e.g. if $r < 4$, $x_{1,2,3\ldots} \le r/4 < 1$, $x_0 > r/4$

2 Dynamics in the logistic map at increasing values of r:

"cobweb diagram"



each plot: 3 sims (different x_0)



dynamics approaches a fixed point x_*

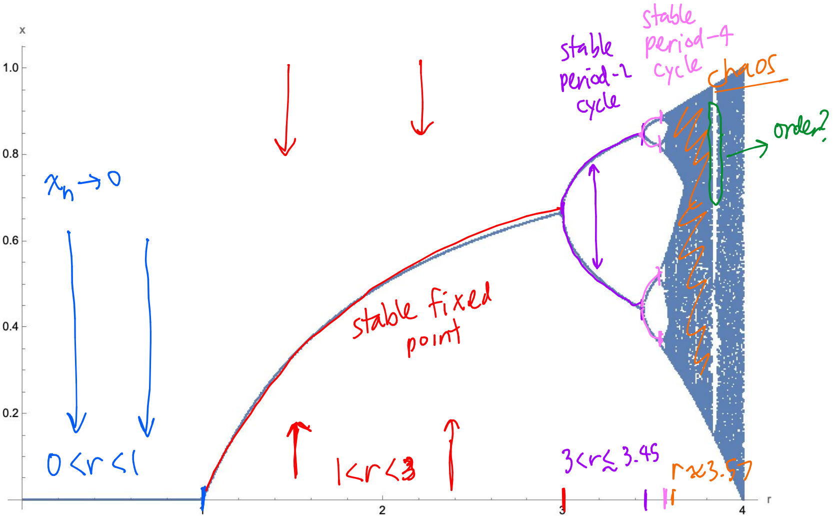
$$x_* = r x_* (1 - x_*)$$

oscillating btwn 2 values

stable period-2 cycle

chaos!

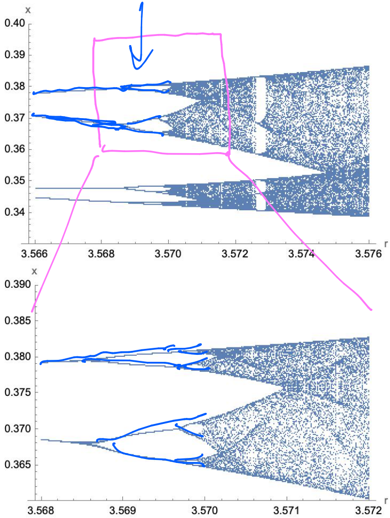
3 A classic picture of the late-time behavior of the logistic map:



In this plot: one choice x_0. Run for many r. Wait for $n=200$.
Plot $\underbrace{x_{201} \ldots x_{400}}$.
typical late-time behavior. "attractor"

4 Zooming in, we see the emergence of period doubling, and self-similar (fractal?) behavior.

period doubling



these 2 pictures look similar

self-similarity

- criticality (phase trans.)

- fractal geometry

- renormalization group