PHYS 5210

Graduate Classical Mechanics

Fall 2022

Lecture 37

Linear stability analysis

November 16

1 Describe the dynamics of the logistic map for $0 < r < 1$.

logistic map: $x_{n+1} = r x_n (1 - x_n)$

$0 \leq r \leq 4$

$0 \leq x_n \leq 1$

Claim: for any x_0, $x_n \to 0$ as $n \to \infty$.

Proof: $\frac{x_{n+1}}{x_n} = r(1 - x_n) \leq 1$. $x_0, x_1, x_2, \ldots$ decreasing

$r<1$, ≤ 1

$\frac{x_{n+1}}{x_n} \leq r < 1$.

$x_0 \leq 1$

$x_1 \leq r$

$x_2 \leq r^2$

$\vdots$

Conclude:

$x_n \leq r^n$.

exponential decay

Hence: 1) $x=0$ is fixed point: $x_n = 0$, then $x_{n+1} = 0$.

If x_* denotes fix pt: $x_* = r x_* (1 - x_*)$

Solved by $x_* = 0$.

2) ... stable fix pt.

Start at $x_0 = \varepsilon > 0$, $x_n \to 0$.

$\boxed{2}$ Describe linear stability analysis for a general 1d map.

Suppose $x_{n+1} = f(x_n)$: x_* is fixed point: $x_* = f(x_*)$.

What if $x_0 = x_* + \delta_0$
perturbatively small.

If define $\delta_n = x_n - x_*$, does δ_n grow large/small?

$x_1 = f(x_0) = x_* + \delta_1 = f(x_* + \delta_0) \approx \underbrace{f(x_*)}_{x_*} + f'(x_*) \cdot \delta_0 + \cdots$

↳ $\delta_1 = f'(x_*) \cdot \delta_0$

Keep going: $\delta_n = f'(x_*)\,\delta_{n-1} = \cdots = \lambda^n \delta_0$, where $\lambda = f'(x_*)$

$\frac{\delta_n}{\delta_0} = \lambda^n$.

stable fix pt when $|\lambda| < 1$ [$\delta_n \to 0$]

If $|\lambda| = 1$: marginal fix pt. [more work]

$\frac{\delta_n}{\delta_0} \to \infty$ if: $|\lambda| > 1$.

unstable

3 Describe the dynamics of the logistic map for $1 < r < 3$.

Back to logistic: $x_{n+1} = r x_n (1 - x_n)$.

Step 1): Find fixed points. $x_* = r x_* (1 - x_*)$

Solved by: $x_* = 0$. AND: $x_* = 1 - \frac{1}{r}$.

Need $r > 1$, so $x_* > 0$.

Step 2): Analyze stability:

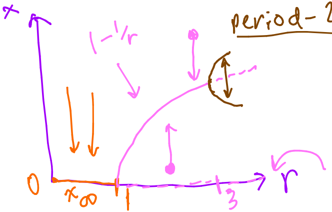
For $x_* = 0$: $f(x) = rx(1-x)$; $\lambda = f'(0) = r(1-0) - r \cdot 0 = r$.

stable if $r < 1$, unstable $r > 1$.

For $x_* = 1 - 1/r$: $\lambda = r \cdot \frac{1}{r} - r(1 - 1/r) = 2 - r$.

stable if $|2 - r| < 1$, or $1 < r < 3$, unstable if $r > 3$.

period-2 cycle for $r > 3$.



unstable fixed pt

- 2 unstable, 0 stable fixed pts
  $\rightarrow$ persistent dynamics.

4 Describe the emergence of a 2-cycle for $r > 3$.

What's stable for $r \approx 3.01$ is period-2 cycle (2-cycle).

$x_{n+2} = x_n$

$x_{n+1} \neq x_n$

repeating pattern (not chaotic)

Define $f_{(2)}(x) = f(f(x))$. So $x_{n+2} = f_{(2)}(x_n)$

$f_{(2)}(x) = r\left[rx(1-x)\right]\left[1 - rx(1-x)\right]$ Quartic !!

Goal: $f_{(2)}$ has (2) stable fixed pts (jump btwn in cycle!)

Look for $x_* = f_{(2)}(x_*)$. 2 solns: $x_*^{old} = f(f(x_*^{old}))$
$= f(x_*^{old})$ ✓

$x_*^{old} = 0,\ 1 - \frac{1}{r}$.

$$0 = \frac{x_* - f_{(2)}(x_*)}{x_*\left[x_* - (1 - \frac{1}{r})\right]}$$

$$\boxed{0 = r^2 x_*^2 - r(1+r)x_* + (1+r)}$$

5 When is this 2-cycle stable?

Using quadratic eq: $\underline{x_*^{new}} = \frac{1+r}{2r} \pm \frac{\sqrt{(r+1)(r-3)}}{2r}$

Check for stability of period-2 cycle of f:

$=$ stability of fixed pt of $f_{(2)}$.

$(f_{(2)})'(x_*) = \dots = 4 + 2r - r^2.$

For what r stable? $1 \geq |4 + 2r - r^2|$

boundary of stability...

$-1 = 4 + 2r - r^2$

$r = \frac{-2 \pm \sqrt{4 + 4 \cdot 5}}{-2}$

$r = 1 \pm \sqrt{6}$

$+1 = 4 + 2r - r^2$

$r = \frac{-2 \pm \sqrt{4 + 4 \cdot 3}}{-2}$

$= 1 \pm 2$

$= 3$ or -1

2 pos solns: $3 < r < \overbrace{1+\sqrt{6}}^{3.449}$

when 2-cycle is stable.