# PHYS 5210

## Graduate Classical Mechanics

## Fall 2022

# Lecture 38

## Renormalization group for period doubling

November 18

## 1 Review period doubling in the logistic map.

Logistic map: $x_{n+1} = r x_n (1 - x_n)$, $0 \le x_n \le 1$, $0 \le r \le 4$

stable fix pt: $x_* = 0$ stable fix pt: $x_* = 1 - 1/r$. stable period-2 cycle


(mostly) chaos

Goal of today:

"heuristic" theory of transition to chaos via period doubling.

**2** Looking at $f$, $f_{(2)}$, $f_{(4)}$, $f_{(8)}$ at $r = 3.55$ reveals self-similarity:

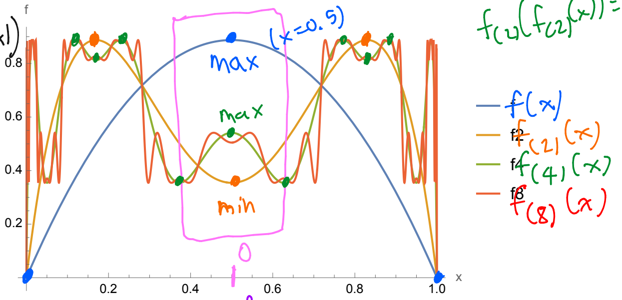
$$f_{(n)}(x) = f\left[f_{(n-1)}(x)\right]$$

$$f_{(1)}(x) = f(x)$$

If $f(x) = r\,x(1-x)$,

$$x_{n+\ell} = f_{(\ell)}(x_n)$$

$$f_{(2)}(f_{(2)}(x)) = f_{(4)}(x)$$



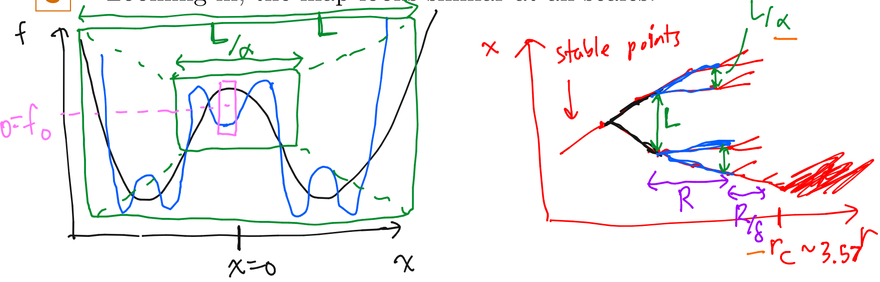

$f_{(\ell)}$ is polynomial of order $2^\ell$

self-similar: Shift $x \to x - 1/2$, then near $x=0$:

$$f_{(2^k)}(x) - f_0 \sim -\frac{1}{\alpha}\left[f_{(2^{k-1})}(\alpha x) - f_0\right]$$

$r = 3.55$ not quite to chaos... aim to work right at onset of chaos

**3** Zooming in, the map looks similar at all scales.




$(\alpha > 1)$

Claim: $\alpha$ & $\delta$ are universal numbers!

Any 1d map $x_{n+1} = f(x_n)$ w/ dissipative, period doubling transition, have same $\alpha/\delta$.

Key idea: $f_{(2^{k+1})}(x) = -\frac{1}{\alpha} f_{(2^k)}(-\alpha x)$

$\approx$ at least near $x=0$ ...

$\boxed{\text{Renormalization group}}$

## 4 The fixed point of RG is a universal function $g(x)$.

Conjecture: the iterated maps (at $r=r_c$), once rescaled, tend to same function. [RG fixed point]

More precisely: for fixed $x$, as $k\to\infty$:

$$(-\alpha)^k f_{(2^k)}\left(\frac{x}{(-\alpha)^k}\right) \to \underline{g(x)} \quad \text{(universal)}$$

independent of $f_{(1)}(x)$.

$$f_{(2^{k+1})}(x) \simeq -\frac{1}{\alpha} f_{(2^k)}(-\alpha x)\ ; \quad \text{or} \quad f_{(2^k)}(x) \approx -\alpha \underbrace{f_{(2^{k+1})}\left(-\frac{x}{\alpha}\right)}$$

$$= -\alpha f_{(2^k)}\left(f_{(2^k)}\left(-\frac{x}{\alpha}\right)\right)$$

$$\cancel{\frac{1}{(-\alpha)^k}}\, g\left((-\alpha)^k x\right) = \frac{-\alpha}{\cancel{(-\alpha)^k}}\, g\left(\cancel{(-\alpha)^k}\cdot \cancel{\frac{1}{(-\alpha)^k}}\, g\left((-\alpha)^k\left(-\frac{x}{\alpha}\right)\right)\right).$$

Define $z = x\cdot(-\alpha)^k$:
$$\boxed{g(z) = -\alpha\, g\left(g\left(-\frac{z}{\alpha}\right)\right)}$$

**5** Estimate $\alpha$ by approximating $g(x)$.

Need to solve: $g(x) = -\alpha\, g\left(g\left(-\frac{x}{\alpha}\right)\right)$, for $g(x)$, $\alpha$.

Note: If $g(x)$ is a sol'n; so is $\lambda \cdot g\left(\frac{x}{\lambda}\right)$, if $\lambda \neq 0$ is const.

$$\cancel{\lambda}\, g\left(\frac{x}{\lambda}\right) = -\alpha \cdot \cancel{\lambda}\, g\left(\frac{1}{\cancel{\lambda}} \cdot \cancel{\lambda}\, g\left(-\frac{x}{\lambda\alpha}\right)\right) : \quad g\left(\frac{x}{\lambda}\right) = -\alpha\, g\left(g\left(-\frac{1}{\alpha}\frac{x}{\lambda}\right)\right)$$

"Choose $\lambda$ efficiently": so that $g(0) = 1$. [if $g(0) \neq 0$].

Try Taylor series: $g(x) = 1 + cx^2 + \ldots$

$$1 + cx^2 = -\alpha\left[1 + c\left[1 + c\left(-\frac{x}{\alpha}\right)^2 + \ldots\right]^2 + \ldots\right]$$

$1 = -\alpha(1+c)$

$1 = -\alpha\left(1 - \frac{\alpha}{2}\right)$, or $0 = 1 + \alpha - \frac{\alpha^2}{2}$

$\alpha \approx 1 + \sqrt{3} \approx 2.73\ldots$

$c = -\alpha c \cdot 2c\left(-\frac{1}{\alpha}\right)^2$, or $1 = -\frac{2c}{\alpha}$

Keep more terms in $g(x)$...

$\alpha \approx 2.502\ldots$